BIE-DML - Discrete Mathematics and Logic

Tutorial 3

Predicate logic

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Winter semester 2024/2025

updated: 14/10/2024, 13:27

3.1 Introduction

3.1.1 Predicate Logic

Predicate logic is an ”upgrade” of propositional logic which allows sophisticated arguments, with the use of objects (constants, variables, functions), their quantification (there exists, for all) and their relationships (predicates).

Definition 3.1. The language of predicate logic consists of: logical symbols and non-logical symbols L:

- symbols for logical connectives $(\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow)$,
- symbols for variables $(k, \ell, m, n, x, y, z, \dots)$,
- symbols for quantifiers, always followed by a variable
  - general – ”for every/any”, ”all”, $(\forall)$
  - existential – ”there is/exists”, ”some”, $(\exists)$
- auxiliary symbols (parentheses),
- symbols for constants $(K, S, \dots)$,
- symbols for predicates $(p, q, r, \dots)$ - of given arity,
- symbols for functions $(f, g, \dots)$ - of given arity.

Definition 3.2. Arity is the number of parameters a function or a predicate takes.
A **term** is a string of symbols which can be obtained by a finite number of the following steps:

1. Any variable or a constant is a term;
2. if $t_1, ..., t_n$ are terms and f is a function symbol of arity n then $f(t_1, ..., t_n)$ is a term.

Definition 3.3. A formula is a sequence of symbols which can be obtained by a finite number of the following steps:

1. If p is a predicate symbol of arity n and $t_1, \dots, t_n$ are terms, then $p(t_1, ..., t_n)$ is a formula. We call it an **atomic formula**.
2. If A, B are formulas then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$ are formulas.
3. If x is a variable and A a formula then $(\forall x)\, A$ and $(\exists x)\, A$ are formulas.

Quantifier negation

Theorem 3.4. *For any formula of predicate logic A, we have that:*

1. $\neg(\exists x)\, A \quad \models\!\!\!\!\dashv \quad (\forall x)\, \neg A$,
2. $\neg(\forall x)\, A \quad \models\!\!\!\!\dashv \quad (\exists x)\, \neg A$.

3.2 Exercises

Exercise 3.1. Analyze this formula of predicate logic:

$$(\forall x)(\exists y)((x < y) \wedge (\forall z)((\exists u)(y = z \cdot u) \Rightarrow ((z = 1) \vee (z = y)))).$$

List all terms, atomic formulas and other subformulas.

Exercise 3.2. Assume we have a unary predicate p, binary predicates $<$ and q, a unary function f, binary functions $+$ and $\cdot$, and constants 1 and 2.

List all terms and atomic formulas for the formulas below.

a) $(\forall x)(p(x) \Rightarrow \neg q(x, y))$

b) $(\forall x)(\forall y)(\exists z)(f(f(x + y)) < z \cdot z + 1)$

c) $(\exists u)(p(u) \wedge (\forall x)(q(u, x) \Rightarrow q(1, x)))$

d) $x \cdot f(f(1)) < (x + x) + x$

e) $1 + 1 = 2$

Exercise 3.3. Formalize the statements below using predicates $h(x), s(x), t(x), w(x)$ with the meaning "x is here", "x is a student", "x is a teacher", "x is writing" respectively. Negate the formulas and translate back into English.

a) "All students are here."

b) "Some students are not here."

c) "Only students are here."

d) "There are no students here."

e) "All students and teachers are here."

f) "All students are here and writing."

g) "Some students are here and writing."

h) "Here are only students that are writing."

Remark 3.5. When we negate $A \wedge B$ the resulting formula $\neg A \vee \neg B$ can be equivalently expressed as $A \Rightarrow \neg B$. The latter can be used to produce a more understandable translation into natural language.

Compare:

- $(\forall x)(\neg s(x) \vee h(x))$ – "Everyone is either not a student or is here"; and
- $(\forall x)(s(x) \Rightarrow h(x))$ – "Everyone who is a student is here", i.e., "All students are here".

Also, compare:

- $(\forall x)(\neg s(x) \vee \neg h(x) \vee \neg w(x))$ – "Everyone is either not a student, not here or not writing"; and
- $(\forall x)((s(x) \wedge h(x)) \Rightarrow \neg w(x))$ – "No students here are writing" (there is another possible equivalent formula and translation, try for yourself).

Exercise 3.4. Express the properties below as formulas in the language of arithmetic on non-negative integers (i.e., we have symbols $+$ denoting addition, $\cdot$ denoting multiplication, $=$ denoting equality, and the set of non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$).

a) $x \leq y$.

b) $x < y$.

c) x is even.

d) x is odd.

e) $x = 0$.

f) $x = 1$.

g) $x = 2$.

h) x is a composite number.

i) x divides y.

We will add new predicates $<, \leq$ and new constants $0, 1, 2$, with their usual meaning.

j) x is a prime number

k) The sum of any two numbers is greater than or equal to both of these numbers.

l) The product of any two numbers greater than 1 is always greater than their sum.

m) There is a least number.

n) There is exactly one least number.

o) There is no greatest number.

We will add a new binary predicate $d(x, y)$ denoting that x divides y.

p) Every divisor must be less than or equal to the number which it divides.

q) x is the greatest common divisor of y and z.

r) x is the least common multiple of y and z.

We will add a new predicate $p(x)$ denoting that x is a prime.

s) There exists a prime number.

t) There are infinitely many primes.

u) Every prime greater than two is odd.

v) There are infinitely many pairs of primes whose difference is 2 (the twin prime conjecture).

Exercise 3.5. Formalise these sentences using the symbols $\{0, 1, 2, +, \cdot, =, <\}$ with their usual meaning on the set of natural numbers. Negate the formulas and translate them back into natural language. You may define new predicates to simplify the formulas.

a) The sum of any two odd numbers is even.

b) No even number is divisible by 3.

c) Exactly one number is equal to its square.

d) There are infinitely many prime numbers.

e) Every pair of numbers has a least common multiple.

f) Every even number greater than 2 can be expressed as the sum of two primes (Goldbach's conjecture).

3.3 More exercises

Exercise 3.6. Formalize the sentences below using a unary predicate $t(x)$ with the meaning "x is a book lying on the table". Negate them and translate the negation back into natural language.

a) "There is (at least) one book on the table."

b) "There is exactly one book on the table."

c) "There are (at least) two books on the table."

d) "There is at most one book on the table."

e) "There are exactly two books on the table."

f) "There are at most two books on the table."

g) "There is only (the book) Lord of the Rings on the table." (Use R as a constant denoting the book "Lord of the Rings").

Exercise 3.7. We will consider unary functions $f(x), x^2$, binary functions $+, \cdot$, constant 0, and binary predicates $=, <$ on the set of real numbers. Formalize the sentences below, negate the formulas and translate them back into natural language.

a) "If f is strictly increasing then $\lim\limits_{x\to\infty} f(x) = \infty$."

b) "The function f has a maximum on a closed interval."

c) * "A quadratic equation has two distinct real roots if the discriminant is greater than 0."

Exercise 3.8. Consider a set of people together with two predicates $f(x)$ meaning "x speaks French" and $s(x)$ meaning "x speaks Spanish". Decide which of the statements below logically follow from the statement "Somebody speaks French and somebody speaks Spanish." You may find the consequences directly or formalize the statements and use known rules.

a) A : "Somebody speaks French or somebody speaks Spanish."

b) B : "Somebody speaks French or Spanish."

c) C : "Somebody speaks French and Spanish."

d) D : "Somebody speaks French or nobody speaks Spanish."

e) E : "Everybody speaks French or Spanish."

Does any statement from a) to e) logically imply the original one (or any of the other statements)?