BIE-DML - Discrete Mathematics and Logic

Tutorial 5

Sets and mappings

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Winter semester 2024/2025

updated: 14/11/2024, 09:52

5.1 Introduction

5.1.1 Sets

In this introduction we recall the basic notions like set definitions, operations etc. We assume that all sets are subsets of one common universe (mostly denoted by $\mathcal{U}$).

Definition 5.1. Let A, B be two subsets of a universe $\mathcal{U}$; let the empty set be denoted by $\emptyset$ or by $\{\}$. Then the property "x is an element of A" is denoted by $x \in A$, and the property "x is not an element of A" is denoted either by $x \notin A$ or $\neg(x \in A)$.

- Set inclusion (subset, $\subseteq$): $(A \subseteq B) \Leftrightarrow (\forall x \in \mathcal{U} : x \in A \Rightarrow x \in B)$.
- Strict set inclusion (proper subset, $\subset$, or $\subsetneqq$): $(A \subsetneqq B) \Leftrightarrow (A \subseteq B \wedge A \neq B)$.
- Set equality: $(A = B) \Leftrightarrow (A \subseteq B \wedge B \subseteq A) \Leftrightarrow (\forall x \in \mathcal{U} : x \in A \Leftrightarrow x \in B)$.

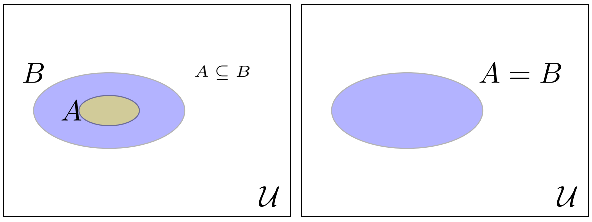

Figure 5.1: Set inclusion in the universe $\mathcal{U}$.

- Intersection of sets A and B: $A \cap B = \{x \in \mathcal{U} : x \in A \wedge x \in B\}$.
- Union of sets A and B: $A \cup B = \{x \in \mathcal{U} : x \in A \vee x \in B\}$.
- Difference of sets A and B: $A \setminus B = \{x \in \mathcal{U} : x \in A \wedge \neg(x \in B)\}$.
- Complement of a set A (in universe $\mathcal{U}$): $\overline{A} = \mathcal{U} \setminus A = \{x \in \mathcal{U} : \neg(x \in A)\}$.

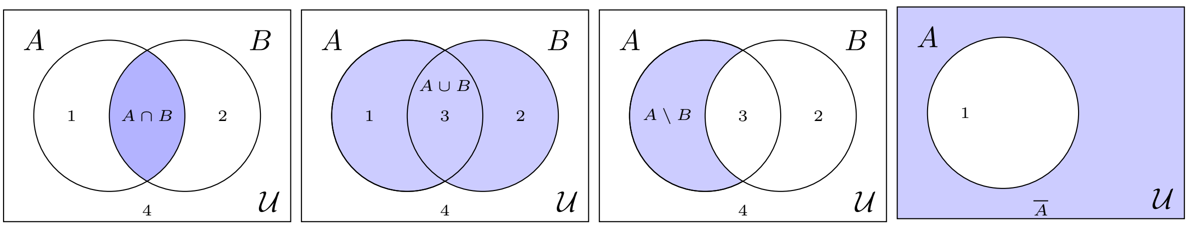

Figure 5.2: Venn diagrams for basic set operations in the universe $\mathcal{U}$.

- Cartesian product of sets A and B (is a subset of the universe $\mathcal{U} \times \mathcal{U}$, but not of $\mathcal{U}$): $A \times B = \{(a, b) \in \mathcal{U} \times \mathcal{U} : a \in A \wedge b \in B\}$.
- Power set $\mathcal{P}(A)$: the set containing all subsets of A, including the empty set $\emptyset$ and A itself, i.e., $\mathcal{P}(A) = \{X \subseteq \mathcal{U} : X \subseteq A\}$.

Remark 5.2. Recall that the operations of intersection, union and Cartesian product can be extended to more than two sets. If $A_1, A_2, A_3, \ldots$ are subsets of the same universe then

$$\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n = \{x \in \mathcal{U} : \forall\, i \in \{1, 2, \ldots, n\}, x \in A_i\}$$

and similarly for $\bigcap\limits_{i=1}^{\infty} A_i$, $\bigcup\limits_{i=1}^{n} A_i$, $\bigcup\limits_{i=1}^{\infty} A_i$.

Furthermore, the Cartesian product of one set is denoted as $A^n = \overbrace{A \times \cdots \times A}^{n\text{-times}}$.

Remark 5.3. Note that the **Cartesian product involving the empty set is always the empty set** which can be seen from the definition:

$$X \times \emptyset = \{(u, v) \in \mathcal{U} \times \mathcal{U} : u \in X \wedge v \in \emptyset\} = \emptyset\,.$$

Remark 5.4. A common mistake is a random swapping of symbols "is an element of" $\in$ with "is a subset of" $\subseteq$, especially when power sets are concerned. The two statements below are equivalent:

$$Y \subseteq X \;\Leftrightarrow\; Y \in \mathcal{P}(X)\,,$$

but after swapping $\in$ with $\subseteq$ we obtain an incorrect notation! Furthermore if we assumed that $X, Y \subseteq \mathcal{U}$ then the statements do not make sense at all ($Y \in X$?? $Y \subseteq P(X)$??).

We will not discuss here all the properties of the above mentioned operations (see lectures). We will focus on verification of set inclusion and set equality. We can either use known set identities or apply one of the following two methods:

1. Universal – using logical formulas: To prove, e.g,. that $A \subseteq B$, we have to prove that every x in the universe $\mathcal{U}$ satisfies

$$x \in A \Rightarrow x \in B.$$

Similarly, we can prove equalities of sets; only here we have equivalences instead of implications. If A or B are composed from other sets, then we systematically (from outer operations to inner ones) replace the respective set operations by relevant logical connectives. To keep **correctness**, we strongly recommend not to use the short version "A" of the formula "$x \in A$".

For instance, let us prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$:

$$\begin{aligned}
\forall\, x \in \mathcal{U} : x \in \overline{A \cup B} &\mathrel{\text{⫤⊨}} \neg(x \in A \cup B)\\
&\mathrel{\text{⫤⊨}} \neg(x \in A \vee x \in B)\\
&\mathrel{\text{⫤⊨}} \neg(x \in A) \wedge \neg(x \in B)\\
&\mathrel{\text{⫤⊨}} x \in \overline{A} \wedge x \in \overline{B}\\
&\mathrel{\text{⫤⊨}} x \in (\overline{A} \cap \overline{B}).
\end{aligned}$$

Using the logical formulas, **tautology** $\top$ or **contradiction** $\perp$ can appear, whose set equivalents are

$$x \in \mathcal{U} \mathrel{\text{⫤⊨}} \top \quad \text{and} \quad x \in \emptyset \mathrel{\text{⫤⊨}} \perp.$$

2. Not universal – Venn diagrams: Another option is to draw Venn diagrams. Every set (let say A) divides the universe into two parts – A and $\overline{A}$ – with the property $A \cup \overline{A} = \mathcal{U}$. Extending this to more sets we obtain a divisions of the universe into disjoint areas characterized by "all the elements from this part belong to the first set, but do not belong to the second set, etc.". To verify inclusion and equality we simply shade the corresponding parts and compare.

For two sets A, B the universe $\mathcal{U}$ can be split into four disjoint areas: $(A \cap \overline{B}), (\overline{A} \cap B), (A \cap B), (\overline{A} \cap \overline{B})$. They are denoted as 1, 2, 3, 4 in the left part of Figure 5.3. Similarly we can construct the diagram for three sets. For more sets n ($n \geq 5$), the picture is less illustrative due to the big amount of areas, since there are 2^n disjoint areas in a diagram! (see the Venn diagram for $n = 4$ on the right of Figure 5.3). Moreover, we cannot use Venn diagrams to verify identities in universes like, e.g., $\mathcal{U} \times \mathcal{U}$. We can only use the projection of Venn diagram into the segment (as we will do, e.g., in Exercise 5.1).

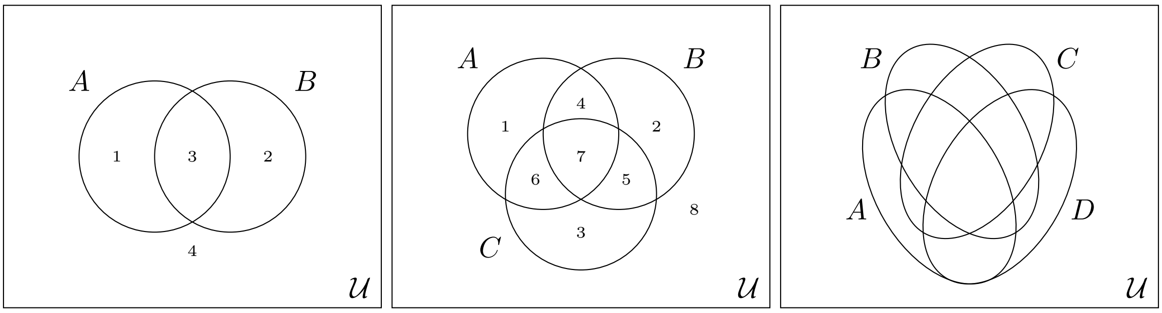

Figure 5.3: Venn diagrams for two, three and four sets in the universe $\mathcal{U}$.

5.1.2 Mappings

We will start this section by recalling some selected definitions related to mappings. In a later chapter we will introduce (and practice sufficiently) binary relations. It is not so hard to see that mappings are a special case of binary relations that satisfy an extra condition.

Definition 5.5. Let A and B be two arbitrary sets. A mapping f from A to B is a subset $f \subseteq A \times B$ such that for every $a \in A$ there is **at most one** $b \in B$ with $(a, b) \in f$. We write $f : A \to B$ and, instead of $(a, b) \in f$, we write $f(a) = b$.

- If $f(a) = b$ for some $a \in A, b \in B$, then a is a **preimage** of b and b is an **image** of a in the mapping f. Every element of A has at most one image (one or none), but every element of B can have any number of preimages in f.

- The **domain** of a mapping f is the set

$$\mathfrak{D}(f) = \{a \in A : \exists b \in B, f(a) = b\} \subseteq A.$$

- The **image** (or **range**) of a mapping f is the set

$$\mathfrak{Im}(f) = \{b \in B : \exists a \in A, f(a) = b\} \subseteq B.$$

  The set B is also called the **codomain** of the function f.

- If not mentioned otherwise, all the mappings in this section will be total, i.e., satisfying $\mathfrak{D}(f) = A$. This means that every element of set A has exactly one image.

- A mapping $f : A \to B$ is **injective** if $\forall x, y \in A : (x \neq y \Rightarrow f(x) \neq f(y))$.

- A mapping $f : A \to B$ is **surjective** if $\forall b \in B\ \exists a \in A : f(a) = b$, i.e., if $\mathfrak{Im}(f) = B$.

- A mapping $f : A \to B$ is **bijective** if it is both surjective and injective.

- Consider two mappings $f : B \to C$ and $g : A \to B$. Then the **composite mapping** $f \circ g : A \to C$ is defined as $(f \circ g)(a) = f(g(a))$ for every $a \in A$.
- We denote by id_M the **identity mapping** on the set M (i.e., the map such that $\forall x \in M, id_M(x) = x$). Let $f : B \to A$. Then a mapping $g : A \to B$ is an **inverse mapping** to f, if $f \circ g = id_A$ and $g \circ f = id_B$.

Remark 5.6. In some books, composition of mappings is defined as $(f \circ g)(x) = g(f(x))$. Don't be confused, just read the definition of the mapping first. The properties presented here can be applied also to this other notation, only the respective sets must be switched carefully.

Finding the domain and the image of a mapping is usually quite simple. Before we present the exercises for this section we will focus a bit on methods of verifying surjectivity, injectivity and bijectivity.

1. Injectivity: By definition, a mapping $f : A \to B$ is injective if

$$\forall\, x, y \in A : x \neq y \Rightarrow f(x) \neq f(y).$$

However, it is much more convenient to work with the equivalent formula

$$\forall\, x, y \in A : f(x) = f(y) \Rightarrow x = y$$

(recall the logical equivalence $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ of propositional logic).

Let us verify, for example, that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^3 + n$ is injective. (This statement is really necessary to verify, the answer "n^3 is injective, n is injective, thus their sum is also injective" is not sufficient. You can see it on the example of the mapping $g(n) = n^3 - n$ which is not injective!)

Consider arbitrary $m, n \in \mathbb{Z}$ and the implication $(f(m) = f(n) \Rightarrow m = n)$. In the next steps we use equivalence (although we need implications only for injectivity), because we will use equivalent steps only:

$$\begin{aligned}
\forall\, m, n \in \mathbb{Z} : \quad f(m) = f(n) &\Leftrightarrow m^3 + m = n^3 + n \\
&\Leftrightarrow m^3 - n^3 + m - n = 0 \\
&\overset{(1)}{\Leftrightarrow} (m-n)(m^2 + mn + n^2) + m - n = 0 \\
&\Leftrightarrow (m-n)(m^2 + mn + n^2 + 1) = 0 \\
&\Leftrightarrow (m - n = 0) \vee (m^2 + mn + n^2 + 1 = 0) \\
&\overset{(2)}{\Leftrightarrow} m = n\,,
\end{aligned}$$

where we used the formula for division of the sum/difference of third powers on line $^{(1)}$, and the fact that equation $m^2 + mn + n^2 + 1 = 0$ has no real solution on line $^{(2)}$. This equation can be solved as a quadratic equation with variable m (and free variable n), whose discriminant is $n^2 - 4(n^2 + 1) < 0$.

As always, if we want to contradict some statement, it is sufficient to find a counterexample. Therefore, if we find example of an element/some elements which does/do not satisfy the considered property, the proof is done.

If we are not sure if injectivity holds or not, a good method is to try to prove it (rather than disprove it). From this process we can usually obtain some ideas about a possible counterexample. Let us illustrate it on the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2 + n$. Let $m, n \in \mathbb{Z}$, then:

$$\begin{aligned} f(m) = f(n) &\Leftrightarrow m^2 + m = n^2 + n \\ &\Leftrightarrow m^2 - n^2 + m - n = 0 \\ &\Leftrightarrow (m-n)(m+n) + (m-n) = 0 \\ &\Leftrightarrow (m-n)(m+n+1) = 0 \\ &\Leftrightarrow (m = n) \vee (m = -n-1) \\ &\not\Leftrightarrow m = n\,, \end{aligned}$$

as we can see, f is not injective because $f(m) = f(n)$ is true not only for $m = n$, but also for any pair of m, n satisfying $m = -n - 1$. Therefore we have not only one, but countably many counterexamples for every $n \in \mathbb{Z}$: $f(n) = f(-n-1)$ and as we can see, $n \neq -n-1$ (because $-\frac{1}{2} \notin \mathbb{Z}$).

It is also sufficient to present one counterexample, such as

$$f(2) = (2^2 + 2) = 6 = ((-3)^2 - 3) = f(-3).$$

2. Surjectivity: A mapping $f : A \to B$ is surjective if every element in B has at least one preimage in f. Our goal is to verify this. More formally:

$$\forall\, b \in B\ \exists\, a \in A : f(a) = b.$$

To do so we have to present an explicit formula for a preimage of any element from B.

To disprove surjectivity, again, we have to present some counterexample, i.e., an element in B which has no preimage. If we cannot see one then we may try to prove surjectivity and obtain some candidates for a counterexample in the process. Let us present the idea on an example using the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(m, n) = (m, m + 2n)$. We want to verify that

$$\forall\, (x, y) \in \mathbb{R}^2\ \exists\, (m, n) \in \mathbb{R}^2 : f(m, n) = (x, y)\,.$$

Consider an arbitrary pair (x, y) in B. We want to find a pair which maps to (x, y) in f, denote it, e.g., (m, n). So we will plug m, n, x, y into the definition of f:

$$\begin{aligned} f(m,n) = (x,y) &\Leftrightarrow (x, y) = (m, m + 2n) \\ &\Leftrightarrow x = m \wedge y = m + 2n \\ &\Leftrightarrow m = x \wedge 2n = y - m \\ &\Leftrightarrow m = x \wedge 2n = y - x \\ &\Leftrightarrow m = x \wedge n = \frac{y - x}{2} \\ &\Leftrightarrow (m, n) = (x, \tfrac{y-x}{2}) \in \mathbb{R}^2\,, \end{aligned}$$

i.e., the mapping f is surjective, because for every $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) = f(\underbrace{x, \tfrac{y-x}{2}}_{\in \mathbb{R}^2})\,.$$

Consider now a mapping$g : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined as $g(m, n) = (m, m + 2n)$ (defined similarly to f but on different sets). The verification of surjectivity can follow the steps of f. However, the interpretation of the result is different, because the preimage of $(m, n) \in \mathbb{Z}^2$, the pair $(m, \frac{n-m}{2})$, does not always lie in $\mathbb{Z}^2$ (if $n - m$ is an odd number). Therefore we can describe all the counterexamples in this way: "No pair $(m, n) \in \mathbb{Z}^2$ for which $n - m$ is odd has a preimage in g". Therefore, g is not surjective.

Remark 5.7. Students are often confused by the names (injective and surjective). Some help could be that both names come from French/Latin. While injective means something in the way "inserting inside", which in a certain sense describes an injective mapping (every element of A is mapped to some of B and no two are mapped to the same one), the prefix "sur-" in the word surjective means "on" in French (images of elements from A are mapped **on**to the whole set B; that's why we also call it "onto").

5.1.3 Cardinality of sets

The **cardinality** of a set A, denoted by $|A|$, is defined as

- the number of elements of A, if A is finite;
- infinite, if A is infinite (there are differnt "kind of infinites").

When considering two sets A, B we can compare their cardinality as follows:

- $|A| = |B|$ if there exists a bijection between A and B;
- $|A| \leq |B|$ if there exists an injection from A to B;
- $|A| < |B|$ if $|A| \leq |B$ and $|A| \neq |B|$.

In particular, when we have an infinite set A, we say that A is **countable** if $|A| = |\mathbb{N}|$, and **uncountable** if it is not countable.

Theorem 5.8. *Let A, B, C be three (finite or infinite) sets. Then*

1) $|A| = |A|$ and $|A| \leq |A|$;

2) if $A \subseteq B$, then $|A| \leq |B|$;

3) if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$;

4) if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$;

5) if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

When dealing with finite sets we can be even more precise.

Theorem 5.9. *Every subset B of a finite set A is finite too.*
Moreover, in this case we have $|B| \leq |A|$, with strict inequality $|B| < |A|$ when $B \subsetneqq A$.

Given two finite sets A, B, we have that $A \cup B$ and $A \times B$ are finite as well.
Moreover, in this case we have $|A \cup B| \leq |A| + |B|$, with equality $|A \cup B| = |A| + |B|$ when the two sets are disjoint, and $|A \times B| = |A| \cdot |B|$.

References:

1. BI-ZDM at FIT:
   https://edux.fit.cvut.cz/courses/BI-ZDM/
2. Selftests and Exercises in Marast at FIT:
   https://marast.fit.cvut.cz/

5.2 Exercises

5.2.1 Sets

Exercise 5.1. Consider the sets X, Y, Z, subsets of the universe $\mathcal{U}$, and their Cartesian products, subsets of $\mathcal{U} \times \mathcal{U}$. Decide whether the following statements are true or not. Prove it, or find a counterexample.

a) $X \cap (Y \setminus Z) = (X \cap Y) \setminus (X \cap Z)$,

b) $X \setminus (Y \cup Z) = (X \setminus Y) \cup Z$,

c) $\overline{X \setminus Y} = \overline{Y \setminus X}$,

d) $\overline{X \cap Y} \subseteq X$,

e) $(X \cap Y) \cup (Y \setminus X) = X$,

f) $(X \setminus Y) \cap (Y \setminus X) = \emptyset$,

g) $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$,

h) $X \times (Y \setminus Z) = (X \times Y) \setminus (X \times Z)$,

i) $\overline{(X \times Y)} = \overline{X} \times \overline{Y}$,

j) $X \setminus (Y \times Z) = (X \setminus Y) \times (X \setminus Z)$,

k) $X \cap (Y \times Z) = (X \cap Y) \times (X \cap Z)$,

l) $X \times \emptyset = \emptyset$.

Exercise 5.2. Let A, B, C be some sets in universe $\mathcal{U}$. Decide if the statements below are true or false. Prove or find a counterexample.

a) $A \setminus B \subseteq A$

b) $A \setminus B = A \cap \overline{B}$

c) $A \cap (B \setminus A) = \emptyset$

d) $(A \setminus B) \cap (B \setminus C) = \emptyset$

e) $(A \setminus B) \setminus C \subseteq A \setminus C$

f) $A \cup (B \setminus A) = A \cup B$

g) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

h) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

i) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

j) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

k) $A \subseteq B$ if and only if $B \subseteq A$

l) $A \subseteq B$ if and only if $A \cap B = A$

m) $A \subseteq B$ if and only if $A \cup B = B$

5.2.2 Mappings

Exercise 5.3. Find the domain $\mathfrak{D}(f)$ and the image $\mathfrak{Im}(f)$ of the following mappings.

a) The mapping f which to every non-negative integer assigns the last digit of its decimal notation.

b) The mapping f which assigns the successor to every non-negative integer.

c) The mapping f which to every binary string assigns its length.

d) The mapping f which to every binary string assigns the number of factors '`01`' included in the string.

e) The mapping f which to every binary string assigns the number of occurrences of `1` minus the number of occurrences of `0` within the string.

f) The mapping f which to every integer k assigns the least square greater than or equal to k.

g) The mapping f which to every pair of real numbers assigns the bigger one of the two.

Exercise 5.4. Decide whether the following mappings are surjective and/or injective. Prove your answer.

a) $f(n) = n + 1, \quad f : \mathbb{Z} \to \mathbb{Z}$.

b) $f(n) = n + 1, \quad f : \mathbb{N}_0 \to \mathbb{N}_0$.

c) $f(n) = 13n, \quad f : \mathbb{Z} \to \mathbb{Z}$.

d) $f(n) = n^3, \quad f : \mathbb{Z} \to \mathbb{Z}$.

e) $f(x) = x^3, \quad f : \mathbb{R} \to \mathbb{R}$.

f) $f(n) = n^2 + 1, \quad f : \mathbb{Z} \to \mathbb{Z}$.

g) $f(n) = (-1)^n \cdot n, \quad f : \mathbb{N}_0 \to \mathbb{Z}$.

h) $f(n) = (n + 1, 2n), \quad f : \mathbb{N}_0 \to \mathbb{N}_0^2$.

i) $f(m, n) = m - n, \quad f : \mathbb{N}_0 \times \mathbb{N} \to \mathbb{Z}$.

Exercise 5.5. Prove or disprove the statements below.

a) If $f \circ g$ is surjective, then f must be surjective. Must g be surjective as well?

b) If $f \circ g$ is injective, then g must be injective. Must f be injective as well?

5.2.3 Cardinality of sets

Exercise 5.6. Let us consider the three sets

$$A = \{1, 2, 3\}, \qquad B = \{3, 4, 5, 6\} \qquad \text{and} \qquad C = \{2, 4, 6\}.$$

Compute the following cardinalities: $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, $|A \cup B|$, $|A \cup C|$, $|B \cup C|$, $|A \cap B \cap C|$, $|A \cup B \cup C|$, $|A \times B|$, $|A \times C|$, $|B \times C|$, $|A \times B \times C|$, $\mathcal{P}(A)$.

Exercise 5.7. Let A, B, C be three sets with cardinality respectively $|A| = 3$, $|B| = 4$, $|C| = 5$. Suppose that $|A \cap B| = 1$, $|B \cap C| = 2$ and $|A \cap C| = 3$ and that only one element belongs to all three sets at the same time. What is the cardinality of the set $A \cup B \cup C$?

Exercise 5.8. Let A, B, C be three finite sets such that

$$|B| = 4,\ |C| = 5,\ |A \cap B| = 3,\ |A \cap C| = 2,\ |B \cap C| = 2,\ |A \cap B \cap C| = 2 \quad \text{and} \quad |A \cup B \cup C| = 9.$$

What is the cardinality of A?

5.3 More exercises

Exercise 5.9. Consider the following sets of non-negative integers defined for every index $i \in \mathbb{N}$ as follows:

$$M_i = \{1, 2, \dots, i\}, \quad N_i = \{i, i+1, i+2, \dots\}.$$

Determine what the following sets look like:

a) $\bigcup_{i=1}^{n} M_i, \quad \bigcup_{i=1}^{\infty} M_i,$

b) $\bigcap_{i=1}^{n} M_i, \quad \bigcap_{i=1}^{\infty} M_i,$

c) $\bigcup_{i=1}^{n} N_i, \quad \bigcup_{i=1}^{\infty} N_i,$

d) $\bigcap_{i=1}^{n} N_i, \quad \bigcap_{i=1}^{\infty} N_i.$

Compute the cardinality of all sets above.

Exercise 5.10. Decide whether the following mappings are surjective and/or injective. Prove or provide a counterexample.

a) $f(m,n) = (m, m \cdot n), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^2;$

b) $f(x,y) = (x+y, x-y), \quad f : \mathbb{R}^2 \to \mathbb{R}^2;$

c) $f(x,y) = (x, x \cdot y), \quad f : \mathbb{R}^2 \to \mathbb{R}^2;$

d) $f(x,y) = (x^3, y^3), \quad f : \mathbb{R}^2 \to \mathbb{R}^2;$

e) $f(x,y) = (\frac{x}{y}, \frac{y}{x}), \quad f : (\mathbb{R}^+)^2 \to (\mathbb{R}^+)^2;$

f) $f(m,n) = (\frac{m}{n}, \frac{n}{m}), \quad f : \mathbb{N}^2 \to \mathbb{R}^2;$

g) $f(m,n) = (m+2n, n-2m), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^2;$

h) $f(m,n) = (m-n, n^2, m+n^2), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^3;$

i) $f(m,n) = (m-n, m^2, m^2+m), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^3;$

j) $f(m,n) = (m-n, m+n, m \cdot n), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^3;$

k) $f(x,y,z) = (2x-z, y^2, x+z), \quad f : \mathbb{R}^3 \to \mathbb{R}^3;$

l) $f(x,y,z) = (2x-z, y^3, x+z), \quad f : \mathbb{R}^3 \to \mathbb{R}^3;$

m) $f(m,n,p) = (m+n, n+p, m+p), \quad f : \mathbb{Z}^3 \to \mathbb{Z}^3;$

n) $f(m,n,p) = (m-n, n-p, m-p), \quad f : \mathbb{Z}^3 \to \mathbb{Z}^3;$

o) $f(m,n,p) = (m+2n, n-p, n+p), \quad f : \mathbb{Z}^3 \to \mathbb{Z}^3;$

p) $f(x,y,z) = (\sqrt{x \cdot y}, \sqrt{x \cdot z}, \sqrt{y \cdot z}), \quad f : (\mathbb{R}^+)^3 \to (\mathbb{R}^+)^3;$

q) $f(x, y, z) = (x + y + z, x + z), \quad f : \mathbb{R}^3 \to \mathbb{R}^2.$

Exercise 5.11 (*)**.** Denote by $\mathcal{A}^*$ the set of all finite words over alphabet $\mathcal{A}$ (including empty word ϵ). Furthermore denote by $\mathcal{A}^n$ the set of all words whose length is exactly n ($n \geq 0$). Decide whether the following mappings are surjective and/or injective. Prove or provide a counterexample.

Note that we strictly distinguish multiplication ($a \cdot b$, always with a dot) from string concatenation (ab, always without a dot).

a) $f(a_1 a_2 \cdots a_m) = m \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{N}_0.$

b) $f(a_1 a_2 \cdots a_m) = a_m a_{m-1} \cdots a_1, \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*.$

c) $f(a_1 a_2 \cdots a_m) = (\mathtt{1} - a_1)(\mathtt{1} - a_2) \cdots (\mathtt{1} - a_m), \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*.$

d) $f(a_1 a_2 \cdots a_n) = a_1 + a_2 + \cdots + a_n, \quad f : \{\mathtt{0},\mathtt{1}\}^n \to \mathbb{N}_0,\ n \geq 1.$

e) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n, \quad f : \{\mathtt{0},\mathtt{1}\}^n \to \mathbb{N}_0,\ n \geq 1.$

f) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n, \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{N}_0.$

g) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n, \quad f : \{\mathtt{0},\mathtt{1},\mathtt{2}\}^n \to \mathbb{N}_0,\ n \geq 1.$

h) $f(a_1 a_2 \cdots a_m) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{m-1} \cdot a_m, \quad f : \{\mathtt{0},\mathtt{1},\mathtt{2}\}^* \to \mathbb{N}_0.$

i) $f(a_1 a_2 \cdots a_m) = a_2 a_3 \cdots a_m a_1, \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*.$

j) $f(a_1 a_2 \cdots a_m) = a_2 a_3 \cdots a_m \mathtt{0}, \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*.$

k) $f(a_1 a_2 \cdots a_m) = a_1 (a_1 \cdot a_2)(a_2 \cdot a_3) \cdots (a_{m-1} \cdot a_m), \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*.$

Exercise 5.12. Prove points 1) to 4) of Theorem 5.8

Exercise 5.13. Let $2\mathbb{Z}$ be the set of even numbers. Prove that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. What is the cardinality of $\mathbb{Z} \times 2\mathbb{Z}$? And the cardinality of $\{\pi\} \times 2\mathbb{Z}$?