Mathematics for Informatics

Constrained optimization, multivariate integration (lecture 3 of 12)

Francesco Dolce

dolcefra@fit.cvut.cz

Czech Technical University in Prague

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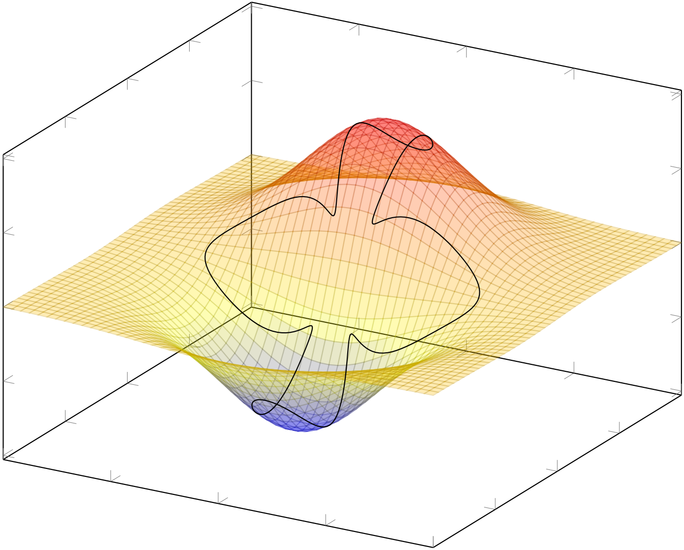
Outline

- Constrained optimization
- Reminder: integration of functions of 1 variable
- 2-variate function integration

Motivation

Find the maximum and minimum points when walking along the black line:



The problem

Let $f : \mathbb{R}^n \to \mathbb{R}$.
Find (local) maxima and minima of f subject to

$$\begin{cases} g_1(x_1, x_2, \ldots, x_n) & = & 0 \\ g_2(x_1, x_2, \ldots, x_n) & = & 0 \\ & \vdots & \\ g_p(x_1, x_2, \ldots, x_n) & = & 0. \end{cases}$$

Set $\mathcal{G} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid g_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, p\}$.

Assumptions

1. The functions f and g_i, with $i = 1, 2, \ldots, p$, have continuous second partial derivatives.

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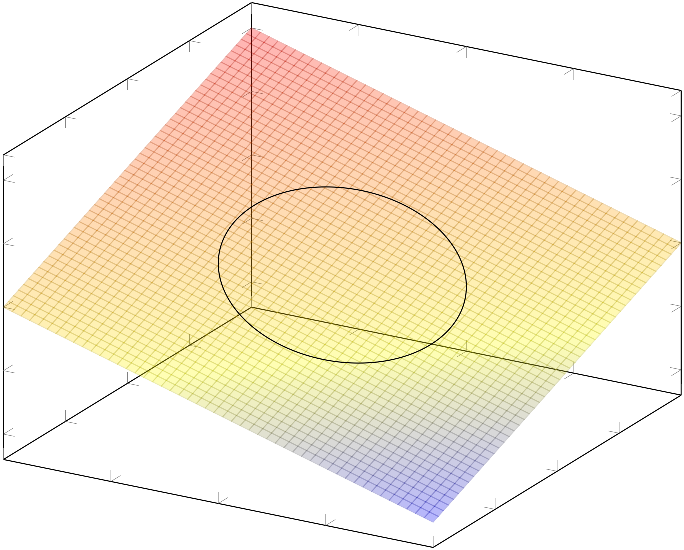
Example

Are the gradients of the following functions linearly independent?

$$g_1(x,y) = 2x + xy^2, \quad g_2(x,y) = 4x + 2xy^2,$$
$$g_3(x,y) = 2xy^2 + 4y^2, \quad g_4(x,y) = 2x + 3xy^2 + 4y^2.$$

Running example

$$f(x,y) = x + y, \qquad \mathcal{G} = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 2\}$$



Necessary condition

Theorem

Assume f has a local extremum in $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{G}$ subject to $\mathcal{G}$.
Then there exist numbers $\mu_1^*, \dots, \mu_p^*$ such that the Lagrangian function L given by

$$L(x_1, \dots, x_n, \mu_1, \dots, \mu_p) = f(x_1, \dots, x_n) + \sum_{i=1}^{p} \mu_i g_i(x_1, \dots, x_n)$$

has zero partial derivatives with respect to $x_1, \dots, x_n$ at the point x^*.

In other words, the following system is true:

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(x^*) + \mu_1^* \dfrac{\partial g_1}{\partial x_1}(x^*) + \cdots + \mu_p^* \dfrac{\partial g_p}{\partial x_1}(x^*) & = & 0 \\ & \vdots & \\ \dfrac{\partial f}{\partial x_n}(x^*) + \mu_1^* \dfrac{\partial g_1}{\partial x_n}(x^*) + \cdots + \mu_p^* \dfrac{\partial g_p}{\partial x_n}(x^*) & = & 0 \end{cases}$$

Sufficient condition

Theorem

Let $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ *and* $\mu^* = (\mu_1^*, \dots, \mu_p^*) \in \mathbb{R}^p$ *such that*

1. *the Lagrangian function* $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$ *has zero partial derivatives with respect to* $x_1, \dots, x_n$ *at the point* $(x^*, \mu^*) \in \mathbb{R}^{n+p}$*;*
2. *the Lagrangian function* $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$ *has zero partial derivatives with respect to* $\mu_1, \dots, \mu_p$ *at the point* $(x^*, \mu^*) \in \mathbb{R}^{n+p}$*;*
3. *for all non-zero* $y \in \mathbb{R}^n$ *satisfying* $y \cdot \nabla g_i(x^*) = 0$ *for* $i = 1, 2, \dots, p$ *we have*

$$y \left(\nabla^2 f(x^*) \; + \; \sum_{i=1}^{p} \mu_i^* \nabla^2 g_i(x^*) \right) y^T > 0.$$

Thus, the function f *has a strict local minimum at* x^* *(subject to* $\mathcal{G}$*).*

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Thus, the function f *has a strict local minimum at* x^* *(subject to* $\mathcal{G}$*).*

If we replace in (iii) the condition "> 0" by "< 0", we obtain a sufficient condition of a strict local maximum.

Example

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Find maxima and minima of $f(x, y) = x + y$ *subject to* $x^2 + y^2 = 2$.

Integration of functions of 1 variable

Let $f : \mathbb{R} \to \mathbb{R}$ and $a < b$.

Recall what does $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ mean, if it exists.

What is its geometrical meaning?

Definition

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The upper Darboux sum of f with respect to the partition Δ is

$$S_{f,\Delta} = \sum_{i=1}^{n} F_{\Delta,i} \cdot (x_i - x_{i-1})$$

and the lower Darboux sum of f with respect to the partition Δ is

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Definition...

The upper Darboux integral (of f over $[a, b]$) is

$$D_f = \inf\{S_{f,\Delta} \colon \Delta \text{ is a partition of } [a, b]\}$$

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If $D_f = d_f$, we call this value the Darboux integral of f over $[a, b]$, and denote it

$$\int_a^b f(x)\,\mathrm{d}x = D_f = d_f.$$

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This is equivalent to the Riemann integral and to Riemann integrability.

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Let f be integrable on $[a, b]$ and on $[b, c]$ (with $a < b < c$).

We have that f is integrable on $[a, c]$ and

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

Primitive function

Let $F(x)$ be a real function which is continuous on $[a, b]$ and differentiable on (a, b).

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$$\forall x \in (a, b), \quad F'(x) = f(x).$$

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Example

Find a primitive function on $(0, 1)$ of the function $f(x) = 2x + x^2$.

Newton's formula

Let f be integrable on $[a, b]$ and $F(x)$ be (one of) its primitive function on (a, b). We have

$$\int_a^b f(x)\,\mathrm{d}x \;=\; [F(x)]_a^b \;=\; F(b) - F(a).$$

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Example

Calculate $\int_0^1 (2x + x^2)\,\mathrm{d}x$.

Substitution

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.

Let φ be a real function differentiable on (α, β) such that φ and φ' are both continuous on $[\alpha, \beta]$.

Let f be continuous on $[\varphi(\alpha), \varphi(\beta)]$ (or if $\varphi(\alpha) \leq \varphi(\beta)$, otherwise continuous on $[\varphi(\beta), \varphi(\alpha)]$).

If $f(\varphi(t))\, \varphi'(t)$ is integrable on $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} f(\varphi(t))\, \varphi'(t)\, \mathrm{d}t = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, \mathrm{d}x.$$

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Example

Calculate $\int_1^2 \frac{2\ln(t)^2}{t}\, \mathrm{d}t$.

Integration by parts

Let f and g be differentiable on (a, b) and let f, g, f', g' be continuous on $[a, b]$. We have

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,\mathrm{d}x.$$

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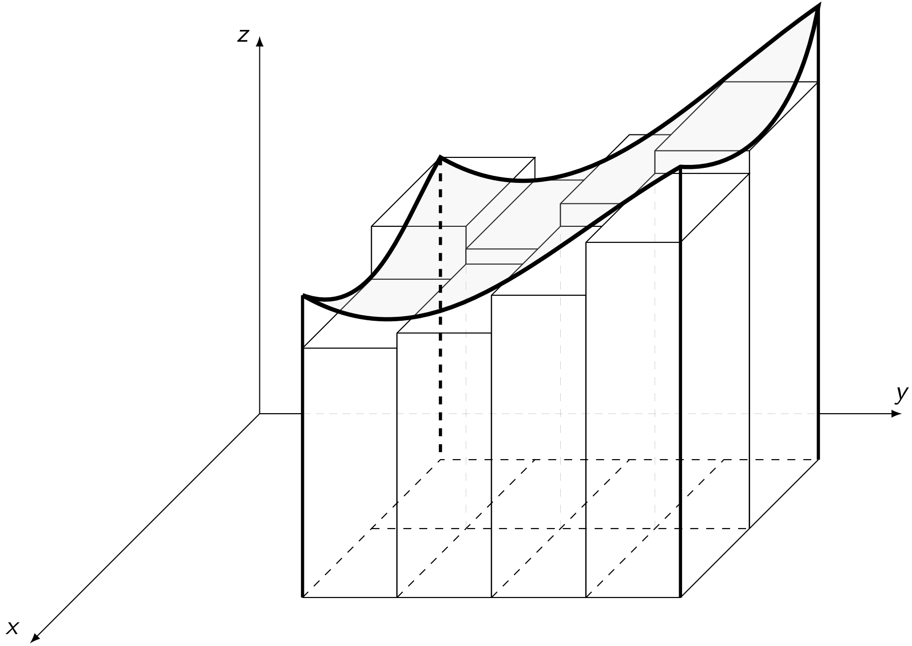
Example

Calculate $\int_1^2 10x \ln x \,\mathrm{d}x$.

2-variate function

Suppose we have a function $f : D \to \mathbb{R}$, where $D = [a, b] \times [c, d]$.

Imagine that this function represents (part of) a surface of some object.
What is the volume of this object?


z
y
x

Definition

Let $\Delta_x = (x_i)_{i=0}^{n}$ define a partition of $[a, b]$ and $\Delta_y = (y_j)_{j=0}^{m}$ a partition of $[c, d]$.

Then, $\Delta = \Delta_x \times \Delta_y$ defines a partitions of $D = [a, b] \times [c, d]$ into rectangles.

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Set

- $F_{\Delta,i,j} = \max\{f(x, y) \colon (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$ and
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while the lower Darboux sum of f with respect to the partition Δ is

$$s_{f,\Delta} = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{\Delta,i,j} \cdot (x_i - x_{i-1}) \cdot (y_j - y_{j-1}).$$

Definition. . .

The upper Darboux integral (of f over D) is

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If $D_f = d_f$, we call this value the (**double**) Darboux integral of f over D, and denote it

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = D_f = d_f.$$

We say that f is (Darboux-)integrable over D.

How to calculate a double integral?

The following statement can be derived from the definition.

Theorem

If f is integrable over $D = [a, b] \times [c, d]$ and one of the integrals

$$\int_a^b \left(\int_c^d f(x, y) \, \mathrm{d}y \right) \mathrm{d}x \quad \textit{or} \quad \int_c^d \left(\int_a^b f(x, y) \, \mathrm{d}x \right) \mathrm{d}y$$

exists, then it is equal to

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Example

Calculate the double integral over $D = [0,2] \times [-1,2]$ of the function $f(x,y) = x^2 y + 1$.

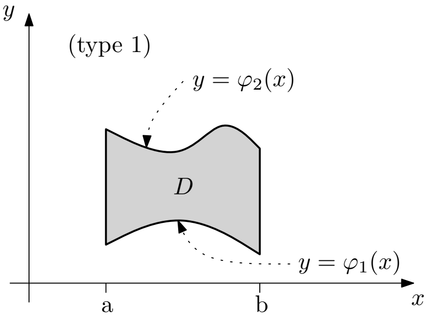
And if D is not a rectangle?

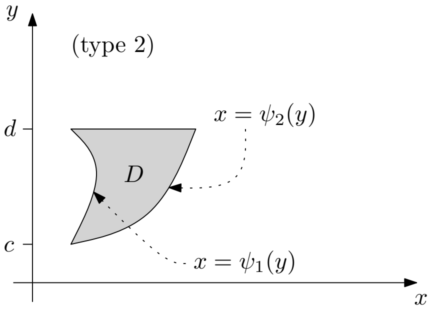
The definition is very similar: we approximate D using smaller and smaller rectangular areas...

Special types of domain D (1/2)

We will consider the following two types of the domain D.

- (type 1) $x \in [a, b]$ and y is bounded by continuous functions $\varphi_1(x)$ and $\varphi_2(x)$;
- (type 2) $y \in [c, d]$ and x is bounded by continuous functions $\psi_1(y)$ and $\psi_2(y)$.





Special types of domain D (2/2)

Double integrals over such D are calculated as follows.

Theorem

If the integral on the right side exists, then we have (for such a domain D):

- *if D is of type 1, then*

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\mathrm{d}y \right) \mathrm{d}x;$$

- *if D is of type 2, then*

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\mathrm{d}x \right) \mathrm{d}y.$$

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Let D be the region given by the triangle with vertices $(0,1)$, $(0,2)$ and $(3,0)$. Calculate

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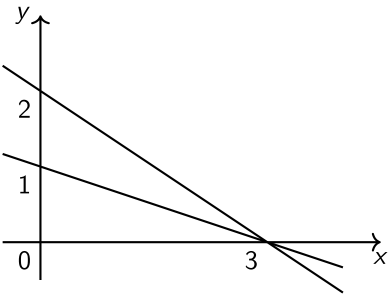

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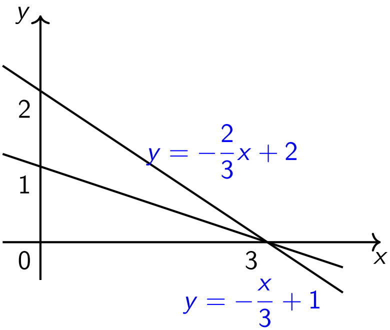

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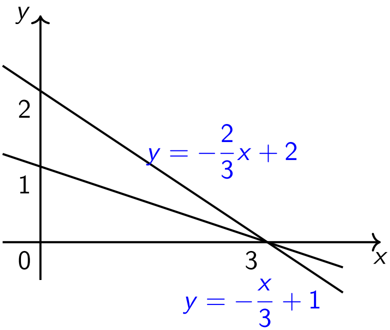

$$\int_0^3 \int_{1-\frac{x}{3}}^{2-\frac{2}{3}x} \frac{x+y}{2}\,\mathrm{d}y\mathrm{d}x = \cdots = \frac{3}{2}$$