Mathematics for Informatics

Algebra: Subgroups, groups generated by a set, cyclic groups (lecture 5 of 12)

Francesco Dolce

dolcefra@fit.cvut.cz

Czech Technical University in Prague

Winter 2024/2025

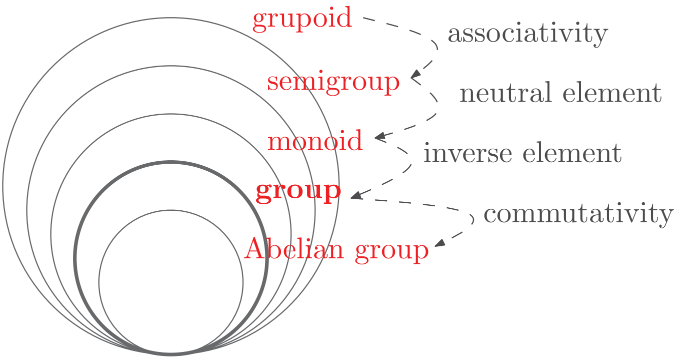
created: September 11, 2024, 14:33

Outline

- Reminder and motivation
- Subgroups
- Groups generated by a set
- Cyclic groups

Reminder of the last lecture

Hierarchy of structures of type “a set and a binary operation”



Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a* ***groupoid***;

Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \dots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a groupoid;*
- *the operation is associative, so it is a* ***semigroup****;*

Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a groupoid;*
- *the operation is associative, so it is a semigroup;*
- *the number* 0 *is the neutral element, so it is a* ***monoid****;*

Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a groupoid;*
- *the operation is associative, so it is a semigroup;*
- *the number* 0 *is the neutral element, so it is a monoid;*
- *the inverse of* $k \neq 0$ *is* $12 - k$ *and the inverse of* 0 *is* 0*, so it is a* ***group****;*

Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a groupoid;*
- *the operation is associative, so it is a semigroup;*
- *the number* 0 *is the neutral element, so it is a monoid;*
- *the inverse of* $k \neq 0$ *is* $12 - k$ *and the inverse of* 0 *is* 0, *so it is a group;*
- *the operation is commutative, thus we have an* ***Abelian group***.

Example (1/4)

Example

Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a groupoid;*
- *the operation is associative, so it is a semigroup;*
- *the number* 0 *is the neutral element, so it is a monoid;*
- *the inverse of* $k \neq 0$ *is* $12 - k$ *and the inverse of* 0 *is* 0, *so it is a group;*
- *the operation is commutative, thus we have an* ***Abelian group***.

Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the set of the residue classes modulo n.

The group $(\mathbb{Z}_n, +_{(\mathrm{mod}\ n)})$ is the additive group modulo n; it is denoted by $\mathbb{Z}_n^+$.

Example (2/4)

Question: Which other set M forms a group with the addition $\pmod{12}$?

Example (2/4)

Question: Which other set M forms a group with the addition $\pmod{12}$?

In order for the operation to be well defined, we must have $M \subset \mathbb{Z}_{12}$:

Question (refined): Which subset of $\mathbb{Z}_{12}$ forms a group with the addition $\pmod{12}$?

Example (2/4)

Question: Which other set M forms a group with the addition $\pmod{12}$?

In order for the operation to be well defined, we must have $M \subset \mathbb{Z}_{12}$:

Question (refined): Which subset of $\mathbb{Z}_{12}$ forms a group with the addition $\pmod{12}$?

Answer: There are quite a lot of them. To find out how to discover them, let us ask this subquestion:

Sub-question: Which is the smallest subset of $\mathbb{Z}_{12}$ that forms a group with addition $\pmod{12}$ and contains the number 2?

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\text{mod } 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\bmod 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\text{mod } 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;
- the operation remains associative, so it is a semigroup;

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\text{mod } 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;
- the operation remains associative, so it is a semigroup;
- 0 remains the neutral element, so it is a monoid;

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\bmod\ 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;
- the operation remains associative, so it is a semigroup;
- 0 remains the neutral element, so it is a monoid;
- each element has its inverse in the set (the set is closed under inversion), so it is a group.

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +_{(\text{mod } 12)})$ is a group:

- M must be closed under addition $\text{mod } 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;
- the operation remains associative, so it is a semigroup;
- 0 remains the neutral element, so it is a monoid;
- each element has its inverse in the set (the set is closed under inversion), so it is a group.

The wanted set is $M = \{0, 2, 4, 6, 8, 10\}$.
We say that M is a subgroup generated by the set $\{2\}$.

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\{2\} \rightarrow \qquad \{0, 2, 4, 6, 8, 10\}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\{0\} \to \quad \{0\}$$

$$\{2\} \to \quad \{0, 2, 4, 6, 8, 10\}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{aligned}
\{0\} &\rightarrow \{0\} \\
\{1\} &\rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \\
\{2\} &\rightarrow \{0, 2, 4, 6, 8, 10\}
\end{aligned}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{aligned}
\{0\} &\rightarrow \{0\} \\
\{1\} &\rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \\
\{2\} &\rightarrow \{0, 2, 4, 6, 8, 10\} \\
\{3\} &\rightarrow \{0, 3, 6, 9\}
\end{aligned}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$
\begin{array}{rc}
\{0\} \rightarrow & \{0\} \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} \\
\{3\} \rightarrow & \{0,3,6,9\} \\
\{4\} \rightarrow & \{0,4,8\}
\end{array}
$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{array}{rc}
\{0\} \rightarrow & \{0\} \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} \\
\{3\} \rightarrow & \{0,3,6,9\} \\
\{4\} \rightarrow & \{0,4,8\} \\
\{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\}
\end{array}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{array}{rc}
\{0\} \rightarrow & \{0\} \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} \\
\{3\} \rightarrow & \{0,3,6,9\} \\
\{4\} \rightarrow & \{0,4,8\} \\
\{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\} \\
\{6\} \rightarrow & \{0,6\}
\end{array}$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$
\begin{array}{lcl}
\{0\} \rightarrow & \{0\} & \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} & \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} & \\
\{3\} \rightarrow & \{0,3,6,9\} & \\
\{4\} \rightarrow & \{0,4,8\} & \\
\{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\} & \leftarrow \{7\} \\
\{6\} \rightarrow & \{0,6\} &
\end{array}
$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$
\begin{array}{lcl}
\{0\} \rightarrow & \{0\} & \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} & \leftarrow \{11\} \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} & \leftarrow \{10\} \\
\{3\} \rightarrow & \{0,3,6,9\} & \leftarrow \{9\} \\
\{4\} \rightarrow & \{0,4,8\} & \leftarrow \{8\} \\
\{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\} & \leftarrow \{7\} \\
\{6\} \rightarrow & \{0,6\} &
\end{array}
$$

Example (4/4)

Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{array}{lcl}
\{0\} \rightarrow & \{0\} & \\
\{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} & \leftarrow \{11\} \\
\{2\} \rightarrow & \{0,2,4,6,8,10\} & \leftarrow \{10\} \\
\{3\} \rightarrow & \{0,3,6,9\} & \leftarrow \{9\} \\
\{4\} \rightarrow & \{0,4,8\} & \leftarrow \{8\} \\
\{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\} & \leftarrow \{7\} \\
\{6\} \rightarrow & \{0,6\} &
\end{array}$$

Back to the original question: there exist 6 different sets $M \subseteq \mathbb{Z}_{12}$ such that $(M, +_{(\text{mod } 12)})$ is a group.

Definition of subgroup

Definition

Let $G = (M, \circ)$ *be a group.*
A *subgroup* *of the group* G *is a pair* $H = (N, \circ)$ *such that:*

- $N \subseteq M$ *and* $N \neq \emptyset$,
- H *is a group.*

Definition of subgroup

Definition

Let $G = (M, \circ)$ be a group.
A subgroup of the group G is a pair $H = (N, \circ)$ such that:

- $N \subseteq M$ *and* $N \neq \emptyset$,
- H *is a group.*

- Idea of substructures with the same properties as the original structure: compare for instance with a subspace of a linear (vector) space.

Definition of subgroup

Definition

Let $G = (M, \circ)$ *be a group.*
A subgroup of the group G *is a pair* $H = (N, \circ)$ *such that:*

- $N \subseteq M$ *and* $N \neq \emptyset$,
- H *is a group.*

- Idea of substructures with the same properties as the original structure: compare for instance with a subspace of a linear (vector) space.

- Similarly, we can define subgroupoids, subsemigroups, submonoids,. . .

Definition of subgroup

Definition

Let $G = (M, \circ)$ *be a group.*
A *subgroup* *of the group* G *is a pair* $H = (N, \circ)$ *such that:*

- $N \subseteq M$ *and* $N \neq \emptyset$,
- H *is a group.*

- Idea of substructures with the same properties as the original structure: compare for instance with a subspace of a linear (vector) space.
- Similarly, we can define subgroupoids, subsemigroups, submonoids,. . .
- A binary operation in the group $G = (M, \circ)$ is a function from $M \times M$ to M. The operation in a subgroup $H = (N, \circ)$ is, to be precise, the restriction of this operation to the set $N \times N$.

Trivial and proper subgroups

In each group $G = (M, \circ)$, there always exist at least two subgroups (if M contains only one element the two coincide):

- the group containing only the neutral element: $(\{e\}, \circ)$, and
- the group itself $G = (M, \circ)$.

These two groups are the trivial subgroups.
Other subgroups are non-trivial or proper subgroups.

Trivial and proper subgroups

In each group $G = (M, \circ)$, there always exist at least two subgroups (if M contains only one element the two coincide):

- the group containing only the neutral element: $(\{e\}, \circ)$, and
- the group itself $G = (M, \circ)$.

These two groups are the trivial subgroups.
Other subgroups are non-trivial or proper subgroups.

Question

If H is a subgroup of a group G, is the neutral element of H identical to the neutral element of G?

Intersection of subgroups

Theorem

Let $H_1, H_2, \ldots, H_n$*, whith* $n \geq 1$*, be subgroups of a group* $G = (M, \circ)$*. Then*

$$H' = \bigcap_{i=1,2,\ldots,n} H_i$$

is also a subgroup of G*.*

Power of an element

Definition

Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as

$$
\begin{aligned}
a^0 &= e \\
a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\
a^{-n} &= (a^{-1})^n = \underbrace{a^{-1} \circ a^{-1} \circ \cdots \circ a^{-1}}_{n \text{ times}}
\end{aligned}
$$

Power of an element

Definition

Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as

$$
\begin{aligned}
a^0 &= e \\
a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\
a^{-n} &= (a^{-1})^n = \underbrace{a^{-1} \circ a^{-1} \circ \cdots \circ a^{-1}}_{n \text{ times}}
\end{aligned}
$$

Note that $a \circ a \circ \cdots \circ a$ can by written without brackets thanks to associativity (for a non-associative operation the result would depend on the order. . .).

Power of an element

Definition

Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as

$$\begin{aligned} a^0 &= e \\ a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\ a^{-n} &= (a^{-1})^n = \underbrace{a^{-1} \circ a^{-1} \circ \cdots \circ a^{-1}}_{n \text{ times}} \end{aligned}$$

Note that $a \circ a \circ \cdots \circ a$ can by written without brackets thanks to associativity (for a non-associative operation the result would depend on the order...).
For all $n, m \in \mathbb{N}$ the following "natural" equalities are true:

- $a^{n+m} = a^n \circ a^m$,
- $a^{nm} = (a^n)^m$.

Power of an element

Definition

Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as

$$\begin{aligned} a^0 &= e \\ a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\ a^{-n} &= (a^{-1})^n = \underbrace{a^{-1} \circ a^{-1} \circ \cdots \circ a^{-1}}_{n \text{ times}} \end{aligned}$$

Note that $a \circ a \circ \cdots \circ a$ can by written without brackets thanks to associativity (for a non-associative operation the result would depend on the order...).

For all $n, m \in \mathbb{N}$ the following "natural" equalities are true:

- $a^{n+m} = a^n \circ a^m$,
- $a^{nm} = (a^n)^m$.

For the additive notation of a group $G = (M, +)$, we define the n-th multiple of the element a and we denote it by $n \times a$ (resp. $-n \times a = n \times (-a)$).

Order of a (sub)group

Definition

The order of a (sub)group $G = (M, \circ)$, denoted $|G|$, is its number of elements. If M is an infinite set, the order is infinite.
According to the order we distinguish between finite and infinite groups.

Order of a (sub)group

Definition

The order of a (sub)group $G = (M, \circ)$, denoted $|G|$, is its number of elements. If M is an infinite set, the order is infinite.
According to the order we distinguish between finite and infinite groups.

Example

The group $\mathbb{Z}_{12}^{+}$ is of order 12. It has 6 subgroups:

- *two trivial: $\{0\}$ and $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$;*
- *and four proper: $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, and $\{0, 2, 4, 6, 8, 10\}$.*

of order 1, 2, 3, 4, 6 and 12.

(Left) cosets of a subgroup

Let G be a group and H be one of its subgroups.
The (left) coset of H in G with respect to an element $g \in G$ is the set

$$gH = \{gh : h \in H\} \quad \text{(or } g + H \text{ in additive notation)}$$

Example

Let us consider the subgroup $H = \{0, 3, 6, 9\}$ of $\mathbb{Z}_{12}$.
Find $g + H$ for all $g \in \mathbb{Z}_{12}$.

(Left) cosets of a subgroup

Let G be a group and H be one of its subgroups.
The (left) coset of H in G with respect to an element $g \in G$ is the set

$$gH = \{gh : h \in H\} \quad \text{(or } g + H \text{ in additive notation)}$$

Example

Let us consider the subgroup $H = \{0, 3, 6, 9\}$ of $\mathbb{Z}_{12}$.
Find $g + H$ for all $g \in \mathbb{Z}_{12}$.

The index of H in G, denoted $[G : H]$, is the number of different cosets of H in G.

Lagrange's Theorem

Theorem

Let H be a subgroup of a finite group G. The order of H divides the order of G.

Lagrange's Theorem

Theorem

Let H be a subgroup of a finite group G. The order of H divides the order of G. More precisely, $|G| = [G : H] \cdot |H|$.

Lagrange's Theorem

Theorem

Let H be a subgroup of a finite group G. The order of H divides the order of G. More precisely, $|G| = [G : H] \cdot |H|$.

This statement connects the abstract structure of a group with divisibility and also with the notion of prime numbers!

Consequence: A group with prime order has only trivial subgroups!

Lagrange's Theorem

Theorem

Let H be a subgroup of a finite group G. The order of H divides the order of G. More precisely, $|G| = [G : H] \cdot |H|$.

This statement connects the abstract structure of a group with divisibility and also with the notion of prime numbers!
Consequence: A group with prime order has only trivial subgroups!

To prove Lagrange's Theorem we need the following lemma.

Lemma

For all $a, b \in G$ one has $|aH| = |bH|$.

Lagrange's Theorem

Theorem

Let H be a subgroup of a finite group G. The order of H divides the order of G. More precisely, $|G| = [G : H] \cdot |H|$.

This statement connects the abstract structure of a group with divisibility and also with the notion of prime numbers!
Consequence: A group with prime order has only trivial subgroups!

To prove Lagrange's Theorem we need the following lemma.

Lemma

For all $a, b \in G$ one has $|aH| = |bH|$.

Question

Let G be a group of order n and $k \in \mathbb{N}$ be such that $k|n$.
Is there any subgroup of G of order k?

Group generated by a set (1/2)

Question: How to find the smallest subgroup of a group $G = (M, \circ)$ containing a given nonempty set $N \subset M$?

Group generated by a set (1/2)

Question: How to find the smallest subgroup of a group $G = (M, \circ)$ containing a given nonempty set $N \subset M$?

Definition

Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The smallest subgroup of G containing N is the subgroup generated by N and is denoted by $\langle N \rangle$.

In particular, for a singleton $N = \{a\}$ we use the notation $\langle a \rangle = \langle \{a\} \rangle$.

Group generated by a set (1/2)

Question: How to find the smallest subgroup of a group $G = (M, \circ)$ containing a given nonempty set $N \subset M$?

Definition

Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The smallest subgroup of G containing N is the subgroup generated by N and is denoted by $\langle N \rangle$.

In particular, for a singleton $N = \{a\}$ we use the notation $\langle a \rangle = \langle \{a\} \rangle$.

Example

For the group $\mathbb{Z}_{12}^{+}$, we have proven that $\langle 2 \rangle = (\{0, 2, 4, 6, 8, 10\}, +_{mod\ 12})$.

Group generated by a set (1/2)

Question: How to find the smallest subgroup of a group $G = (M, \circ)$ containing a given nonempty set $N \subset M$?

Definition

Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The smallest subgroup of G containing N is the subgroup generated by N and is denoted by $\langle N \rangle$.

In particular, for a singleton $N = \{a\}$ we use the notation $\langle a \rangle = \langle \{a\} \rangle$.

Example

For the group $\mathbb{Z}_{12}^{+}$, we have proven that $\langle 2 \rangle = (\{0, 2, 4, 6, 8, 10\}, +_{mod\ 12})$.

Definition

If for a set M it holds that $\langle M \rangle = G$, we say that M is a generating set of G.

Group generated by a set (2/2)

Example

The group $\mathbb{Z}_{12}^{+}$ *is generated, for instance, by the sets* $\{1\}$ *and* $\{5\}$*, i.e.*

$$\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_{12}^{+}.$$

Group generated by a set (2/2)

Example

The group $\mathbb{Z}_{12}^{+}$ *is generated, for instance, by the sets* $\{1\}$ *and* $\{5\}$*, i.e.*

$$\langle 1\rangle = \langle 5\rangle = \mathbb{Z}_{12}^{+}.$$

Theorem

Let $G = (M, \circ)$ *be a group and* $N \subset M$ *a nonempty set. The following holds:*

- *the subgroup* $\langle N\rangle$ *equals the intersection of all subgroups containing* N*, i.e.*

$$\langle N\rangle = \bigcap\{H\colon H \text{ is a subgroup of } G \text{ containing } N\}$$

Group generated by a set (2/2)

Example

The group $\mathbb{Z}_{12}^{+}$ *is generated, for instance, by the sets* $\{1\}$ *and* $\{5\}$*, i.e.*

$$\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_{12}^{+}.$$

Theorem

Let $G = (M, \circ)$ *be a group and* $N \subset M$ *a nonempty set. The following holds:*

- *the subgroup* $\langle N \rangle$ *equals the intersection of all subgroups containing* N*, i.e.*

$$\langle N \rangle = \bigcap \{H \colon H \text{ is a subgroup of } G \text{ containing } N\}$$

- *all elements in* $\langle N \rangle$ *can be obtained by means of "group span", i.e.,*

$$\left\{ a_1^{k_1} \circ a_2^{k_2} \circ \cdots a_n^{k_n} \ : \ n \in \mathbb{N},\ a_i \in N,\ k_i \in \mathbb{Z} \right\}.$$

Group generated by a set (2/2)

Example

The group $\mathbb{Z}_{12}^{+}$ *is generated, for instance, by the sets* $\{1\}$ *and* $\{5\}$*, i.e.*

$$\langle 1\rangle = \langle 5\rangle = \mathbb{Z}_{12}^{+}.$$

Theorem

Let $G = (M, \circ)$ *be a group and* $N \subset M$ *a nonempty set. The following holds:*

- *the subgroup* $\langle N\rangle$ *equals the intersection of all subgroups containing* N*, i.e.*

$$\langle N\rangle = \bigcap\{H\colon H \text{ is a subgroup of } G \text{ containing } N\}$$

- *all elements in* $\langle N\rangle$ *can be obtained by means of "group span", i.e.,*

$$\left\{a_1^{k_1} \circ a_2^{k_2} \circ \cdots a_n^{k_n} \;:\; n \in \mathbb{N},\ a_i \in N,\ k_i \in \mathbb{Z}\right\}.$$

Groups generated by one element (1/2)

We have seen that the additive group $\mathbb{Z}_{12}^{+}$ is equal to $\langle 1 \rangle$, $\langle 5 \rangle$, $\langle 7 \rangle$, and $\langle 11 \rangle$.

The following theorem generalize this fact.

Theorem

An additive group modulo n is equal to $\langle k \rangle$ if and only if k and n are coprimes.

Groups generated by one element (1/2)

We have seen that the additive group $\mathbb{Z}_{12}^{+}$ is equal to $\langle 1 \rangle$, $\langle 5 \rangle$, $\langle 7 \rangle$, and $\langle 11 \rangle$.

The following theorem generalize this fact.

Theorem

An additive group modulo n is equal to $\langle k \rangle$ if and only if k and n are coprimes.

Proof.

This statement is a consequence of a general theorem which will be proven later and of the fact that $\mathbb{Z}_{n}^{+} = \langle 1 \rangle$ for all $n \geq 2$. □

Groups generated by one element (2/2)

The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\mathrm{mod}\ p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^\times$.

Groups generated by one element (2/2)

The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\text{mod } p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^\times$.

Example

Is there a one-element set generating the group $\mathbb{Z}_{11}^\times$*?*

Groups generated by one element (2/2)

The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\text{mod } p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^\times$.

Example

Is there a one-element set generating the group $\mathbb{Z}_{11}^\times$*?*

Yes, for example $\langle 2 \rangle = \mathbb{Z}_{11}^\times$.

Groups generated by one element (2/2)

The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\text{mod } p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^\times$.

Example

Is there a one-element set generating the group $\mathbb{Z}_{11}^\times$?

Yes, for example $\langle 2 \rangle = \mathbb{Z}_{11}^\times$.

On the other hand, $\langle 3 \rangle = (\{1, 3, 4, 5, 9\}, \cdot_{(mod\ 11)})$.

Groups generated by one element (2/2)

The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\text{mod } p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^{\times}$.

Example

Is there a one-element set generating the group $\mathbb{Z}_{11}^{\times}$?

Yes, for example $\langle 2 \rangle = \mathbb{Z}_{11}^{\times}$.

On the other hand, $\langle 3 \rangle = (\{1, 3, 4, 5, 9\}, \cdot_{(mod\ 11)})$.

Finding the generator(s) of a multiplicative group $\mathbb{Z}_p^{\times}$ is more complicated than for an additive group $\mathbb{Z}_n^{+}$.
Multiplicative groups have more complicated and interesting structure.

Definition of cyclic group

Definition

A group $G = (M, \circ)$ *is cyclic if there exists an element* $a \in M$ *such that* $\langle a \rangle = G$*.*
This element is a generator of the cyclic group.

Definition of cyclic group

Definition

A group $G = (M, \circ)$ is cyclic if there exists an element $a \in M$ such that $\langle a \rangle = G$. This element is a generator of the cyclic group.

- $\mathbb{Z}_n^+$ is a cyclic group for every n and its generators are all positive numbers $k \leq n$ coprime with n.

Definition of cyclic group

Definition

A group $G = (M, \circ)$ is cyclic if there exists an element $a \in M$ such that $\langle a \rangle = G$. This element is a generator of the cyclic group.

- $\mathbb{Z}_n^+$ is a cyclic group for every n and its generators are all positive numbers $k \leq n$ coprime with n.
- The infinite group $(\mathbb{Z}, +)$ is cyclic and it has just two generators: 1 and -1.

Definition of cyclic group

Definition

A group $G = (M, \circ)$ is cyclic if there exists an element $a \in M$ such that $\langle a \rangle = G$. This element is a generator of the cyclic group.

- $\mathbb{Z}_n^+$ is a cyclic group for every n and its generators are all positive numbers $k \leq n$ coprime with n.
- The infinite group $(\mathbb{Z}, +)$ is cyclic and it has just two generators: 1 and -1.
- $\mathbb{Z}_{11}^\times$ is cyclic, and 2 is a generator.

Why "cyclic"?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

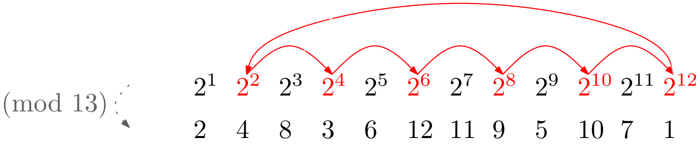
$\pmod{13}$

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
2	4	8	3	6	12	11	9	5	10	7	1

Why "cyclic"?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

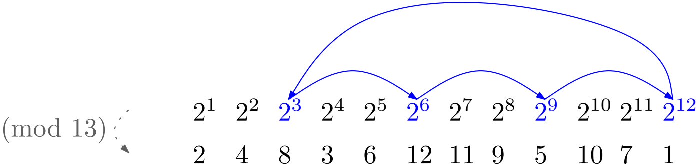


subgroups: $\{1, 3, 4, 9, 10, 12\}$

Why “cyclic”?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.



subgroups: $\{1, 3, 4, 9, 10, 12\}$,$\{1, 5, 8, 12\}$

Why “cyclic”?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

$\pmod{13}$	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
	2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1,3,4,9,10,12\}$,$\{1,5,8,12\}$,$\{1,3,9\}$

Why "cyclic"?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2,\quad 2^2 = 4,\quad 2^3 = 8,\quad 2^4 = 3, \ldots,\quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$\pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1,3,4,9,10,12\}$, $\{1,5,8,12\}$, $\{1,3,9\}$, $\{1,12\}$.

Why "cyclic"?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

	2^1	~~2^2~~	~~2^3~~	~~2^4~~	2^5	~~2^6~~	2^7	~~2^8~~	~~2^9~~	~~2^{10}~~	2^{11}	~~2^{12}~~
$\pmod{13}$	2	~~4~~	~~8~~	~~3~~	6	~~12~~	11	~~9~~	~~5~~	~~10~~	7	~~1~~

subgroups: $\{1,3,4,9,10,12\}$, $\{1,5,8,12\}$, $\{1,3,9\}$, $\{1,12\}$.
generators: 2, 6, 7, 11.

Fermat's Theorem (1/2)

Theorem

In a cyclic group $G = (M, \circ)$ *of order* n*, for all elements* $a \in M$*, it holds that*

$$a^n = e$$

Where e *is the neutral element of* G*.*

Fermat's Theorem (1/2)

Theorem

In a cyclic group $G = (M, \circ)$ of order n, for all elements $a \in M$, it holds that

$$a^n = e$$

Where e is the neutral element of G.

Proof.

Consider a sequence of elements from M: $a, a^2, a^3, a^4, \ldots$

Fermat's Theorem (1/2)

Theorem

In a cyclic group $G = (M, \circ)$ of order n, for all elements $a \in M$, it holds that

$$a^n = e$$

Where e is the neutral element of G.

Proof.

Consider a sequence of elements from M: $a, a^2, a^3, a^4, \ldots$
Denote q the smallest number such that $a^q = e$. Clearly $q \leq n$ (why?!)

Fermat's Theorem (1/2)

Theorem

In a cyclic group $G = (M, \circ)$ of order n, for all elements $a \in M$, it holds that

$$a^n = e$$

Where e is the neutral element of G.

Proof.

Consider a sequence of elements from M: $a, a^2, a^3, a^4, \dots$
Denote q the smallest number such that $a^q = e$. Clearly $q \leq n$ (why?!)
The set $a, a^2, \cdots, a^q$ is the subgroup $\langle a \rangle$ and has order q.
By Lagrange's Theorem, we have that q divides n, i.e,. there exists $k \in \mathbb{N}$ such that $n = qk$.

Fermat's Theorem (1/2)

Theorem

In a cyclic group $G = (M, \circ)$ of order n, for all elements $a \in M$, it holds that

$$a^n = e$$

Where e is the neutral element of G.

Proof.

Consider a sequence of elements from M: $a, a^2, a^3, a^4, \dots$
Denote q the smallest number such that $a^q = e$. Clearly $q \leq n$ (why?!)
The set $a, a^2, \cdots, a^q$ is the subgroup $\langle a \rangle$ and has order q.
By Lagrange's Theorem, we have that q divides n, i.e,. there exists $k \in \mathbb{N}$ such that $n = qk$.
We have $a^n = a^{qk} = (a^q)^k = e^k = e$. □

Fermat's Theorem (2/2)

$\mathbb{Z}_p^\times$ is always a cyclic group (it is not trivial to prove it) and its order is $p - 1$.

Fermat's Theorem (2/2)

$\mathbb{Z}_p^\times$ is always a cyclic group (it is not trivial to prove it) and its order is $p-1$.

Applying the previous theorem to $\mathbb{Z}_p^\times$ we obtain the well-known Fermat's Little Theorem.

Corollary (Fermat's Little Theorem)

For an arbitrary prime number p and an arbitrary $1 \leq a < p$ we have that

$$a^{p-1} \equiv 1 \, (mod \; p).$$

How to find all generators (1/2)

Generally, to find all generators is not an easy task (e.g., in groups $\mathbb{Z}_p^\times$ we are not able to do it algorithmically); but if we have one, it is easy to find all the others.

Theorem

If $(G, \circ)$ is a cyclic group of order n and a is one of its generator, then a^k is a generator if and only if k and n are coprime.

How to find all generators (1/2)

Generally, to find all generators is not an easy task (e.g., in groups $\mathbb{Z}_p^\times$ we are not able to do it algorithmically); but if we have one, it is easy to find all the others.

Theorem

If $(G, \circ)$ is a cyclic group of order n and a is one of its generator, then a^k is a generator if and only if k and n are coprime.

To prove the previous theorem we use the following

Lemma

Let $D = \{mk + \ell n \mid m, \ell \in \mathbb{Z}\}$.
Then $\gcd(k, n) = \min\{|a| \mid a \in D \setminus \{0\}\}$.

How to find all generators (2/2)

Corollary

In a cyclic group of order n, the number of all generators is equal to $\varphi(n)$.

Where φ is the Euler's (totient) function, which assigns to each integer n the number of integers less than n that are coprime with n

How to find all generators (2/2)

Corollary

In a cyclic group of order n, the number of all generators is equal to $\varphi(n)$.

Where φ is the Euler's (totient) function, which assigns to each integer n the number of integers less than n that are coprime with n

$\mathbb{Z}_p^\times$ is a cyclic group of order $p-1$ and thus it has $\varphi(p-1)$ generators.

How to find all generators (2/2)

Corollary

In a cyclic group of order n, the number of all generators is equal to $\varphi(n)$.

Where φ is the Euler's (totient) function, which assigns to each integer n the number of integers less than n that are coprime with n

$\mathbb{Z}_p^\times$ is a cyclic group of order $p-1$ and thus it has $\varphi(p-1)$ generators.

An effective algorithm for evaluating $\varphi(n)$ does not exist; if it existed, we would be able to find the integer factorization of arbitrarily large n and RSA would not be safe!

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$\pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$\pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1, 3, 4, 9, 10, 12\}$

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

(mod 13)

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1, 3, 4, 9, 10, 12\}$, $\{1, 5, 8, 12\}$

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$\pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1,3,4,9,10,12\}$, $\{1,5,8,12\}$, $\{1,3,9\}$

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

(mod 13)	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
	2	4	8	3	6	12	11	9	5	10	7	1

subgroups: $\{1, 3, 4, 9, 10, 12\}$, $\{1, 5, 8, 12\}$, $\{1, 3, 9\}$, $\{1, 12\}$.

Subgroups of cyclic group are cyclic

Theorem

Any subgroup of a cyclic group is again a cyclic group.

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

(mod 13)												
	2^1	~~2^2~~	~~2^3~~	~~2^4~~	2^5	~~2^6~~	2^7	~~2^8~~	~~2^9~~	~~2^{10}~~	2^{11}	~~2^{12}~~
	2	~~4~~	~~8~~	~~3~~	6	~~12~~	11	~~9~~	~~5~~	~~10~~	7	~~1~~

subgroups: $\{1, 3, 4, 9, 10, 12\}$, $\{1, 5, 8, 12\}$, $\{1, 3, 9\}$, $\{1, 12\}$.
generators: 2, 6, 7, 11.

Order of an element

Let G be a group and $g \in G$.
The order of g (in G) is the order of the group that is generated by g.

In the finite case, we have the equivalence $\mathrm{order}(g) = \#\langle g \rangle$.

Order of an element

Let G be a group and $g \in G$.
The order of g (in G) is the order of the group that is generated by g.

In the finite case, we have the equivalence $\mathrm{order}(g) = \#\langle g \rangle$.

Example

Find the order of all elements in $\mathbb{Z}_5^\times$ *and in* $\mathbb{Z}_7^\times$.