# Mathematics for Informatics

## Algebra: Homomorphisms, permutations (lecture 6 of 12)

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# The same groups and distinct elements (1/5)

| $\mathbb{Z}_5^\times$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 1 | 3 |
| 3 | 3 | 1 | 4 | 2 |
| 4 | 4 | 3 | 2 | 1 |

| $\mathbb{Z}_4^+$ | 0 | 1 | 2 | 3 |
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inverse elements: $$\begin{aligned} 1^{-1} &= 1, & 2^{-1} &= 3, \\ 3^{-1} &= 2, & 4^{-1} &= 4. \end{aligned}$$

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Aren't these two groups in fact the same group differing only in the "names" of their elements?

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- The neutral element has very special and unique properties: we rename 1 to 0.

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Let us try to rename the elements of the group $\mathbb{Z}_5^\times$ so to get $\mathbb{Z}_4^+$:

- The neutral element has very special and unique properties: we rename $1$ to $0$.
- If the complete structure should be preserved, then the only two-elements subgroup $\{1, 4\}$ (in $\mathbb{Z}_5^\times$) must correspond to the subgroup $\{0, 2\}$ (in $\mathbb{Z}_4^+$): we map $4 \leftrightarrow 2$.

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- Now, it remains to rename only $2$ and $3$; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.

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- Now, it remains to rename only $2$ and $3$; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.
- It suffices to reorder the rows... and we have the Cayley table of $\mathbb{Z}_4^+$.

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We have found a way to rename the elements in one table to gain an exact copy of the other table (after rearranging rows and columns).

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This renaming is actually an **injective** mapping of the set $\{1, 2, 3, 4\}$ **onto** the set $\{0, 1, 2, 3\}$; let us denote it $\varphi_1$:

$$\varphi_1(1) = 0, \qquad \varphi_1(2) = 3, \qquad \varphi_1(3) = 1, \qquad \varphi_1(4) = 2.$$

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We have pointed out that the mapping $\varphi_2$ works as well:

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Would all bijections do the same job? And if not, what makes these two so special?

# The same groups and distinct elements (4/5)

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|---|---|---|---|---|
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The resulting table is not the Cayley table of the group $\mathbb{Z}_4^+$, because, e.g., $3 + 3 \pmod 4 \neq 1$.

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The resulting table is not the Cayley table of the group $\mathbb{Z}_4^+$, because, e.g., $3 + 3 \pmod 4 \neq 1$.

The bijection $\varphi_3$ does not give rise to the same structure of the group $\mathbb{Z}_4^+$; only $\varphi_1$ and $\varphi_2$ have this property.

# The same groups and distinct elements (5/5)

The desired property, which only the bijections $\varphi_1$ and $\varphi_2$ have, is the following:

$$\text{for all } n, m \in \{1, 2, 3, 4\}, \text{ we have } \varphi(n \times_5 m) = \varphi(n) +_4 \varphi(m),$$

where $\times_5$ denotes the operation in the group $\mathbb{Z}_5^\times$, and $+_4$ the one in the group $\mathbb{Z}_4^+$.

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> *In words: If we apply the operation $\times_5$ to two arbitrary elements of the group $\mathbb{Z}_5^\times$ and then we send the result to $\mathbb{Z}_4^+$ by $\varphi$, we obtain the same result as when we first transform by $\varphi$ the elements to $\mathbb{Z}_4^+$ and **then** apply the operation $+_4$.*

$$\begin{tikzcd}
Z_4^+ \quad Z_5^\times & n, m \arrow[dl, bend right, "\varphi"'] \arrow[dr, bend left, "\times_5"] & \\
\varphi(n), \varphi(m) \arrow[dr, bend right, "+_4"'] & & n \times_5 m \arrow[dl, bend left, "\varphi"] \\
& \varphi(n) +_4 \varphi(m) = \varphi(n \times_5 m) &
\end{tikzcd}$$

# Homomorphism and isomorphism

## Definition

*Let $G = (M, \circ_G)$ and $H = (N, \circ_H)$ be two groupoids. The mapping $\varphi : M \to N$ is a homomorphism from $G$ to $H$ if*

$$\text{for all } x, y \in M, \text{ we have } \quad \varphi(x \circ_G y) = \varphi(x) \circ_H \varphi(y).$$

*If, moreover, $\varphi$ is injective (resp. surjective, resp. bijective) we say that $\varphi$ is a monomorphism (resp. epimorphism, resp. isomorphism).*

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A homomorphism preserves the structure given by the binary operation: the result is the same if we first apply the operation and then the homomorphism or if we proceed inversely.

The only thing needed to define a homomorphism is that the set is closed under the binary operation; this is why we have defined homomorphism for the most general structures, i.e., groupoids.

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Isomorphic groups have the same order.

# Fundamental properties of homomorphisms (1/2)

## Theorem

*Let* $\varphi$ *be a homomorphism from a group* $G = (M, \circ_G)$ *to a group* $H = (N, \circ_H)$*. The group* $\varphi(G) = (\varphi(M), \circ_H)$ *is a subgroup of* $H$*.*

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- For all $x, y, z \in M$ we have that

$$(\varphi(x) \circ_H \varphi(y)) \circ_H \varphi(z) = \varphi(x \circ_G y) \circ_H \varphi(z) = \varphi((x \circ_G y) \circ_G z) =$$
$$= \varphi(x \circ_G (y \circ_G z)) = \varphi(x) \circ_H \varphi(y \circ_G z) = \varphi(x) \circ_H (\varphi(y) \circ_H \varphi(z))$$

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- It can be shown similarly that the inverse of $\varphi(x)$ is $\varphi(x^{-1})$. □

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**Consequences of the previous theorem and its proof:**

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## Example

$$\begin{aligned} \varphi : \mathbb{Z}_4^+ &\rightarrow \mathbb{Z}_8^+ \\ n &\mapsto 2n \end{aligned}$$

*is a homomorphism and* $\varphi(\mathbb{Z}_4^+)$ *is the subgroup* $\{0, 2, 4, 6\} \leq \mathbb{Z}_8^+$.

# . . . up to isomorphism (1/2)

Isomorphic groups are in fact identical, they differ only in the names of their elements (as we have seen in the case of groups $\mathbb{Z}_4^+$ and $\mathbb{Z}_5^\times$).
If we say that there exists one group with a certain property up to isomorphism, it means that all groups with this property are isomorphic to each other.
We prove three well-known statements of this kind.

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**Proof: hint.**

Let $G = \langle a \rangle$ be a cyclic group with generator $a$.
We show that an arbitrary infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, and that an arbitrary cyclic group of order $n$ is isomorphic to $\mathbb{Z}_n^+$.
The rest follows from the transitivity of the relation "to be isomorphic". $\square$

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The rest follows from the transitivity of the relation "to be isomorphic". □

$(\mathbb{Z}, +)$ and $\mathbb{Z}_n^+$ are the only cyclic groups up to isomorphism.

# . . . up to isomorphism (2/2)

The Klein group is the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \circ)$, where

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$$

and $\circ$ is the component-wise addition modulo $2$: e.g., $(1,0) \circ (1,1) = (0,1)$.

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The Klein group is not cyclic and thus cannot be isomorphic to $\mathbb{Z}_4^+$!
It is possible to show this (try it, it is easy):

**Theorem**

*There exists only two groups of order $4$ which are not isomorphic.*

$\mathbb{Z}_4^+$ and the Klein group are the only two groups of order $4$ up to isomorphism.

# Permutations (1/2)

The symmetric group $\mathcal{S}_n$ of the set of all permutations over $\{1, 2, 3, \ldots, n\}$ with the operation of composition.

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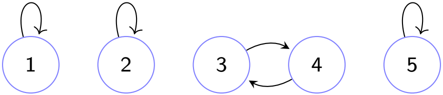

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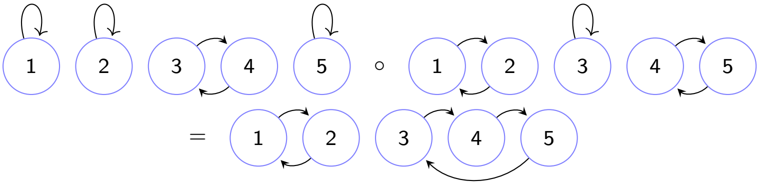

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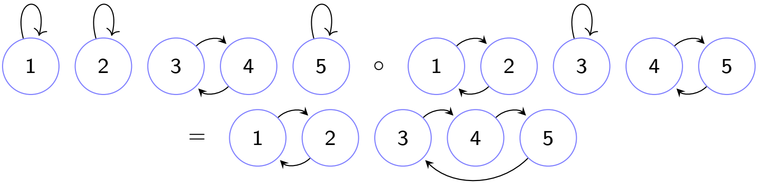


- The composition of permutations is associative, the permutation $(1\ 2\ 3\ \cdots n)$ is the neutral element, and the inverse element is the inverse permutation. Hence, $\mathcal{S}_n$ is a group of order $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

# Group of permutations

Subgroups of the symmetric group $\mathcal{S}_n$ are called groups of permutations.

**Example**

*The permutation* $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ *swapping the elements* $3$ *and* $4$ *generates a subgroup of* $\mathcal{S}_5$ *containing two elements:* $(1\ 2\ 4\ 3\ 5)$ *and* $(1\ 2\ 3\ 4\ 5)$.

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The structure of the subgroups of $\mathcal{S}_n$ is very (in some sense maximally) rich:

**Theorem (Cayley)**

*Each finite group is isomorphic to some group of permutations.*

**Proof: hint only for interested.**

Let $a$ be an element of a group $G$ of order $n$ with a binary operation $\circ$.
Put $\pi_a(x) = a \circ x$. Since in any group we can divide uniquely, $\pi_a$ is a bijection and thus a permutation. The desired monomorphism is the mapping defined for each element $a$ in this way: $\varphi(a) = \pi_a$. $\square$