Mathematics for Informatics

Introduction to fuzzy logic
(lecture 8 of 12)

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Introduction

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Sometimes we want to describe systems by properties which are not evaluated as `true` or `false` (and we do not have the exact value of x).

Fuzzy logic / fuzzy control systems allow such description: truth and falsehood notions are graded and allow to state, for instance, that the water is "**tepid**".

Universe and crisp sets

Let U denote the universe, that is, our playground containing every set that we may consider.

A set $A \subset U$ can be given by its **characteristic function**:

$$\chi_A : U \to \{0,1\}, \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

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There is a bijection between sets and characteristic functions, so we identify each set with its characteristic function.

A is a set in the ordinary sense, sometimes called a crisp set.

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A **fuzzy subset** A of a set X is a function $\mu_A : X \to [0, 1]$.

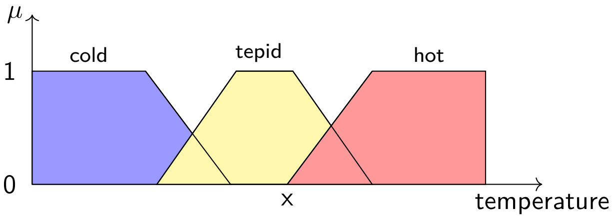
For every element $x \in X$, the **degree of membership** of x to A is given by $\mu_A(x) \in [0, 1]$.

Example

Let $X = [0, 100]$ be the set of temperatures of water in our pot.

We consider three fuzzy subsets of X to describe **cold**, **tepid** and **hot** temperatures.

The membership functions may be given as follows:



Operations on crisp sets

Given a set X and its power set $\mathcal{P}(X)$ (the set of all subsets of X), the operations of **union**, **intersection**, and **complement** are given as follows (for usual sets):

$$A \cup B = \{x : \ x \in A \text{ or } x \in B\},$$
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$$\chi_{A \cup B} = \max\{\chi_A, \chi_B\},$$
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Operations on fuzzy sets

For fuzzy sets, we can define the membership function of a union, intersection, or a complement in the same way.

Let A and B be two *fuzzy* subsets of X.
We set

$$\mu_{A\cup B} = \max\{\mu_A, \mu_B\},$$
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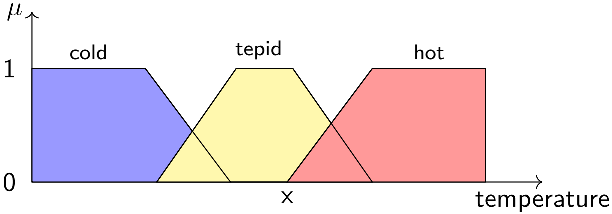

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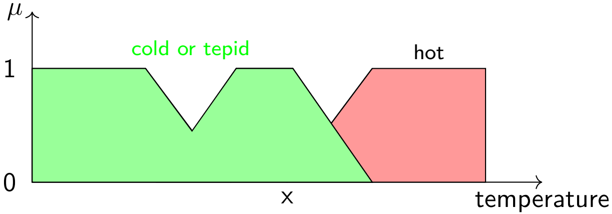

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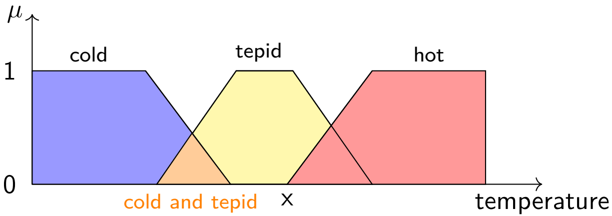

Operations revisited

Our choice for fuzzy set operation was fast.
Let A and B be two subsets of X. We have

$$\begin{aligned}\chi_{A\cap B}(x) &= \min\{\chi_A(x), \chi_B(x)\} \\ &= \chi_A(x)\cdot\chi_B(x) \\ &= \max\{0,\ \chi_A(x)+\chi_B(x)-1\}.\end{aligned}$$

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We can extend the second definition to membership functions and obtain another definition of union and intersection of fuzzy sets.
We shall do this in a more general fashion.

t-norms

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$$\star : [0, 1] \times [0, 1] \to [0, 1]$$

that would interpret intersection of two fuzzy sets.

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4. $(x \star y) \star z = x \star (y \star z)$ for all $x, y, z \in [0,1]$ (*associativity*),
5. $x \leq y$ and $w \leq z$ implies $x \star w \leq y \star z$ (*monotonicity*).

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The distinct t-norms give us distinct strategies on how to interpret intersection of fuzzy sets.

If we have intersection and complement, we define union by $A \cup B = \left(A^{\complement} \cap B^{\complement}\right)^{\complement}$ (De Morgan's laws).

Reasoning in fuzzy logic

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In fuzzy logic, to interpret such implications, we consider "the water is cold" and "my shower is bad" as fuzzy sets and we decide using an **implication** function

$$[0,1] \times [0,1] \to [0,1].$$

This is sometimes called approximate reasoning.

Implication

An **implication** is a function $I : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions for all $x, y, z \in [0,1]$:

1. If $x \leq z$, then $I(x,y) \geq I(z,y)$;
2. if $y \leq z$, then $I(x,y) \leq I(x,z)$;
3. $I(0,y) = 1$;
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Examples:

(i) Mamdani: $I(x,y) = \min\{x,y\}$ (this fails item 3, but usually in knowledge systems we are not interested in rules where the antecedent part is false),

(ii) Willmott: $I(x,y) = \max\{1-x, \min\{x,y\}\}$,

(iii) ...

Standard fuzzy logic controllers

A controller measures some inputs and gives an output following some rules.
For instance, we have the following set of rules:

- r1. If "water is cold", then "shower is bad".
- r2. If "water is tepid", then "shower is good".
- r3. If "water is hot", then "shower is bad".

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1. Measure the input variables, i.e., the temperature $x_0 \in X$.
2. Transform the measured values into fuzzy sets: we have fuzzy sets with constant membership functions $\mu_{cold}(x_0)$, $\mu_{tepid}(x_0)$, and $\mu_{hot}(x_0)$.
3. Apply all the rules: we obtain 3 *control* fuzzy sets
   - $\mu_{r_1}(y) = I(\mu_{cold}(x_0), \mu_{bad}(y))$,
   - $\mu_{r_2}(y) = I(\mu_{tepid}(x_0), \mu_{good}(y))$,
   - $\mu_{r_3}(y) = I(\mu_{hot}(x_0), \mu_{bad}(y))$.
4. Aggregate the control fuzzy sets into one fuzzy set C.
5. Defuzzify C to obtain the output value $c \in Y$.

Standard fuzzy logic controllers

For each step, there are many possible choices:

- transformation to fuzzy sets (and application of operators to construct the antecedent of all implications in the rules);
- implication itself;
- aggregation;
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A usual choice is Gödel t-norm and Mamdani for implication, union for aggregation, and a defuzzification by center of gravity:

$$y_0 = \frac{\int_Y y\mu_C(y)\mathrm{d}y}{\int_Y \mu_C(y)\mathrm{d}y}$$

(or replace by sums if Y is discrete).