

NIE-MPI: Tutorial 5

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5.1 Subgroups

Exercise 5.1. Which of the following sets forms a subgroup of the group $(\mathbb{Q} \setminus \{0\}, \cdot)$?

- (a) The set of all even numbers without zero;
- (b) The set of all odd numbers;
- (c) $\{2^n : n \in \mathbb{Z}\}$;
- (d) $\{2^n \cdot 3^m : n, m \in \mathbb{Z}\}$;
- (e) $\left\{\frac{1+2n}{1+2m} : n, m \in \mathbb{Z}\right\}$.

Exercise 5.2. Find some other subgroup(s) of the group $(\mathbb{Q} \setminus \{0\}, \cdot)$ distinct from the ones in the previous exercise.

Exercise 5.3. Find all subgroups of the group given by following Cayley table:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Hint: when looking for a subgroup, be more specific on what subgroup you look for.

Exercise 5.4. Is $G = (M, \cdot)$ with $M = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a \neq 0 \vee b \neq 0\}$ a subgroup of the group $(\mathbb{R} \setminus \{0\}, \cdot)$?

Exercise 5.5. Specify the following subgroups of $(\mathbb{Z}, +)$.

- (a) $\langle 2 \rangle$;
- (b) $\langle 5 \rangle$;
- (c) $\langle \{2, 3\} \rangle$;
- (d) $\langle \{2, 4\} \rangle$;
- (e) $\langle \{6, 12\} \rangle$;
- (f) $\langle \{n, m, \ell\} \rangle$ for $n, m, \ell \in \mathbb{N}^+$.

5.2 Cyclic groups and generators

Exercise 5.6. Find all generators and all subgroups of $\mathbb{Z}_{11}^\times = (\mathbb{Z}_{11} \setminus \{0\}, \cdot_{11})$.

Exercise 5.7. Find all generators and all subgroups of $\mathbb{Z}_{13}^\times = (\mathbb{Z}_{13} \setminus \{0\}, \cdot_{13})$. Find the inverse elements of 12, 5, and 11.

Exercise 5.8. What is the probability that an element of \mathbb{Z}_{23}^\times chosen randomly is a generator?

Exercise 5.9. Is the number 5 a generator of the group \mathbb{Z}_{23}^\times ?

What is the least number of “computation steps” needed to decide this question?