Interval estimation of parameters

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

Suppose we observe a **random sample** $X_1, \dots, X_n$ (independent and identically distributed random variables) from an **unknown distribution**. We aim to estimate:

- The **shape** of the distribution – its type and parametric family.
- The **parameters** of the distribution.

To get a graphical overview of the shape of the distribution, we can find:

- The **histogram**, which is an approximation of the **density**.
- The **empirical distribution function**, which estimates the real **distribution function**.

Most often we aim to estimate the expectation $\mathrm{E}\, X_i = \mu$ and the variance $\operatorname{var} X_i = \sigma^2$. We have found **unbiased** and **consistent** estimators as:

- The **sample mean** as the estimator for the expectation:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

- The **sample variance** as the estimator for the variance:

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2.$$

Confidence intervals

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Definition

Let $X_1, \ldots, X_n$ be a random sample from a distribution with a parameter θ. The interval (L, U) with boundaries given by statistics $L \equiv L(\mathbf{X}) \equiv L(X_1, \ldots, X_n)$ and $U \equiv U(\mathbf{X}) \equiv U(X_1, \ldots, X_n)$ fulfilling

$$\mathrm{P}\big(L < \theta < U\big) = 1 - \alpha$$

is called the $100 \cdot (1 - \alpha)\%$ **confidence interval** for θ.

Statistics L and U are called the **lower** and **upper** bound of the confidence interval.

The number $(1 - \alpha)$ is called **confidence level**.

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- For a **symmetric** or **two-sided** interval we choose L and U such that

$$\mathrm{P}(\theta < L) = \frac{\alpha}{2} \quad \text{and} \quad \mathrm{P}(U < \theta) = \frac{\alpha}{2}.$$

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- The most common values are $\alpha = 0.05$ and $\alpha = 0.01$, i.e., the ones that gives a 95% confidence interval or a 99% confidence interval.

One-sided confidence intervals

If we are interested only in a lower or upper bound, we construct statistics L or U such that

$$\mathrm{P}\left(L < \theta\right) = 1 - \alpha \quad \text{or} \quad \mathrm{P}\left(\theta < U\right) = 1 - \alpha.$$

This means that

$$\mathrm{P}\left(\theta < L\right) = \alpha \quad \text{or} \quad \mathrm{P}\left(U < \theta\right) = \alpha,$$

and intervals $(L, +\infty)$ or $(-\infty, U)$ are called the upper or lower **confidence intervals**, respectively.

In this case we speak about **one-sided confidence intervals**.

Construction of confidence intervals

There are several possible ways how to construct confidence intervals, depending on the underlying distribution and meaning of estimated parameters. We will use the following approach:

- Find a statistics $H(\theta)$, which:
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The statistics $H(\theta)$ is often chosen using the distribution of a point estimate of the parameter θ, i.e., sample mean for the expectation or sample variance for the theoretical variance.

Confidence intervals for the expectation

If the variance σ^2 is known:

Theorem

Suppose we have a random sample $X_1, \ldots, X_n$ from the normal distribution $\mathsf{N}(\mu, \sigma^2)$ and suppose that we know the value of σ^2. The two-sided symmetric $100 \cdot (1-\alpha)\%$ confidence interval for μ is

$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \, , \, \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right),$$

*where $z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$ is the **critical value** of the standard normal distribution, i.e., such a number for which it holds that $\mathrm{P}(Z > z_{\alpha/2}) = \alpha/2$ for $Z \sim \mathsf{N}(0,1)$.*

The One-sided $100 \cdot (1-\alpha)\%$ confidence intervals for μ are then

$$\left(\bar{X}_n - z_{\alpha} \frac{\sigma}{\sqrt{n}} \, , \, +\infty \right) \quad \text{and} \quad \left(-\infty \, , \, \bar{X}_n + z_{\alpha} \frac{\sigma}{\sqrt{n}} \right),$$

using the same notation.

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The moment generating function of the normal distribution with parameters μ and σ^2 is:

$$M_X(s) = \mathrm{E}[e^{sX}] = \int_{-\infty}^{+\infty} e^{sx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x$$

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The moment generating function of a sum of independent random variables is the product of their generating functions.

The moment generating function of a sum of i.i.d. normal variables is:

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\begin{aligned}
M_{\text{sum}}(s) &= \mathrm{E}\left[e^{s\sum_{i=1}^{n} X_i}\right] = \mathrm{E}[e^{sX_1} \cdot \dots \cdot e^{sX_n}] \overset{\text{independence}}{=} \mathrm{E}[e^{sX_1}] \cdot \dots \cdot \mathrm{E}[e^{sX_n}] \\
&= \prod_{i=1}^{n} M_i(s) \overset{\text{identical distribution}}{=} (M(s))^n \\
&= \left(e^{\mu s - \frac{\sigma^2 s^2}{2}}\right)^n = e^{n\mu s - \frac{n\sigma^2 s^2}{2}}.
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Comparing with the moment generating function of one normal variable we see that the generating function of the sum corresponds with the normal distribution $\mathrm{N}(n\mu, n\sigma^2)$.
[to continue] □

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Thus $\sum_{i=1}^{n} X_i \sim \mathrm{N}(n\mu, n\sigma^2)$ and therefore $\bar{X}_n \sim \mathrm{N}\left(\mu, \frac{n\sigma^2}{n^2}\right) = \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

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$$= \mathrm{P}\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X}_n - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \mathrm{P}\left(z_{\alpha/2}\frac{\sigma}{\sqrt{n}} > \mu - \bar{X}_n > -z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

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1 - \alpha &= \mathrm{P}(z_{1-\alpha/2} < Z < z_{\alpha/2}) = \mathrm{P}\left(-z_{\alpha/2} < \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) \\
&= \mathrm{P}\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X}_n - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \mathrm{P}\left(z_{\alpha/2}\frac{\sigma}{\sqrt{n}} > \mu - \bar{X}_n > -z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= \mathrm{P}\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu - \bar{X}_n < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)
\end{aligned}$$

$\square$

Confidence intervals for the expectation

If the variance σ^2 is known:

Proof

Thus $\sum_{i=1}^{n} X_i \sim \mathrm{N}(n\mu, n\sigma^2)$ and therefore $\bar{X}_n \sim \mathrm{N}\left(\mu, \frac{n\sigma^2}{n^2}\right) = \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.
Thus after standardization we have

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1).$$

From the definition of the **critical value** $z_{\alpha/2}$ we have $\mathrm{P}(Z > z_{\alpha/2}) = \alpha/2$.
It follows that $\mathrm{P}(Z < z_{\alpha/2}) = 1 - \mathrm{P}(Z > z_{\alpha/2}) = 1 - \alpha/2$. It means that

$$\mathrm{P}(z_{1-\alpha/2} < Z < z_{\alpha/2}) = \mathrm{P}(Z < z_{\alpha/2}) - \mathrm{P}(Z < z_{1-\alpha/2}) = 1 - \alpha/2 - (1 - 1 + \alpha/2) = 1 - \alpha.$$

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$$\begin{aligned}
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□

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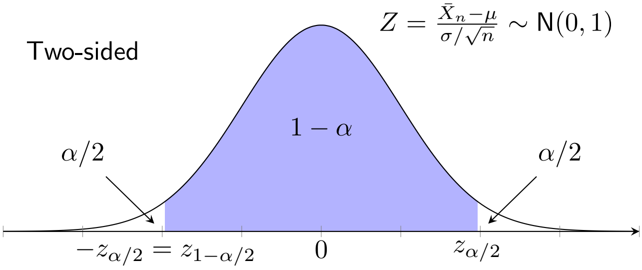

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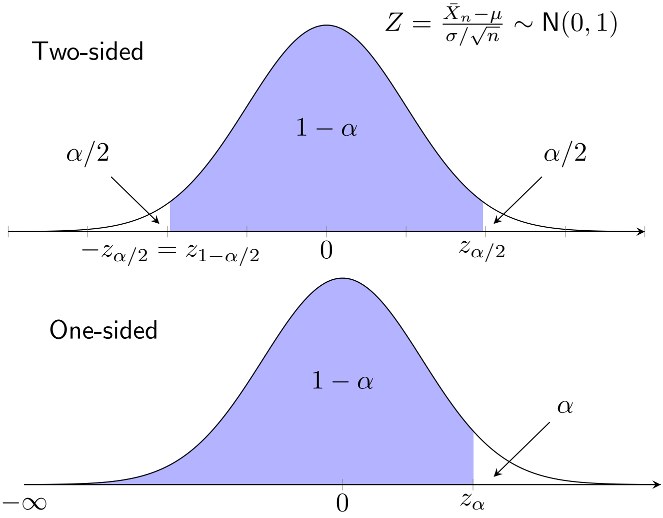

Confidence intervals for the expectation

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This fact can be utilized to form confidence intervals also for other than normal distributions.

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where $z_{\alpha/2}$ is the critical value of $\mathrm{N}(0,1)$. The one-sided confidence intervals are constructed analogously.

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- The **approximate** confidence level of such intervals $\mathrm{P}\left(\mu \in (\cdots)\right)$ is then $1-\alpha$.

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- The **approximate** confidence level of such intervals $\mathrm{P}\left(\mu \in (\cdots)\right)$ is then $1-\alpha$.
- *Large enough* usually means $n = 30$ or $n = 50$. For some distributions which are further away from the normal distribution (e.g., not unimodal, skewed), n must be even larger.

Confidence intervals for the expectation

If the variance σ^2 is unknown:

Most often in practice we do not know the variance σ^2, but only have the observed data at our disposal.

As seen last time, the variance can be estimated using the sample variance

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

We will now show how to utilize the sample variance and adjust the intervals so that the confidence level would be exactly $1 - \alpha$.

Chi-square and Student’s t-distribution

We use the following new distributions:

Definition

Suppose we have a random sample $Y_1, \dots, Y_n$ from the normal distribution $\mathrm{N}(0,1)$. Then we say that the random variable

$$Y = \sum_{i=1}^{n} Y_i^2$$

has the ***chi-square*** *(χ^2) distribution with n degrees of freedom.*

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Definition

Suppose we have a random sample $Y_1, \dots, Y_n$ from $\mathrm{N}(0,1)$, $Y = \sum_{i=1}^{n} Y_i^2$ and an independent variable Z also from $\mathrm{N}(0,1)$. Then we say that the random variable

$$T = \frac{Z}{\sqrt{Y/n}}$$

has the ***Student's t****-distribution with n degrees of freedom.*

The critical values for both distributions can be found in tables.

Chi-square distribution and the variance

We estimate the unknown variance σ^2 using the sample variance

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

The distribution of the sample variance is connected with the chi-square distribution:

Theorem

Suppose we have a random sample $X_1, \ldots, X_n$ from the normal distribution $\mathrm{N}(\mu, \sigma^2)$. Then

$$\frac{(n-1)s_n^2}{\sigma^2}$$

*has the **chi-square** distribution with $n-1$ degrees of freedom.*

Proof

See literature. □

Student's t-distribution and the expectation

The distribution of the sample mean with σ replaced by $s_n = \sqrt{s_n^2}$ is connected with the t-distribution:

Theorem

Suppose we have a random sample $X_1, \ldots, X_n$ from the normal distribution $\mathrm{N}(\mu, \sigma^2)$*. Then*

$$T = \frac{\bar{X}_n - \mu}{s_n / \sqrt{n}}$$

has the ***Student's t****-distribution with $n-1$ degrees of freedom.*

Proof

We can rewrite T as:

$$T = \frac{\bar{X}_n - \mu}{\sqrt{s_n^2/n}} = \frac{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s_n^2}{\sigma^2(n-1)}}}.$$

The numerator has standard normal distribution $\mathrm{N}(0,1)$, under the square root in the denominator we have χ^2_{n-1} divided by $(n-1)$. The distributions of $\bar{X}_n$ and s_n^2 are independent (see literature), thus the whole fraction has indeed the t_{n-1} distribution. $\square$

Confidence intervals for the expectation

If the variance σ^2 is unknown:

If the variance σ^2 is unknown we estimate the σ by taking the square root of the sample variance $s_n = \sqrt{s_n^2}$. Standardization of $\bar{X}_n$ with s_n leads to the **Student's t-distribution**:

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The one-sided $100 \cdot (1-\alpha)\%$ confidence intervals for μ are

$$\left(\bar{X}_n - t_{\alpha,n-1} \frac{s_n}{\sqrt{n}} \,,\; +\infty \right) \quad \text{and} \quad \left(-\infty \,,\; \bar{X}_n + t_{\alpha,n-1} \frac{s_n}{\sqrt{n}} \right)$$

using the same notation.

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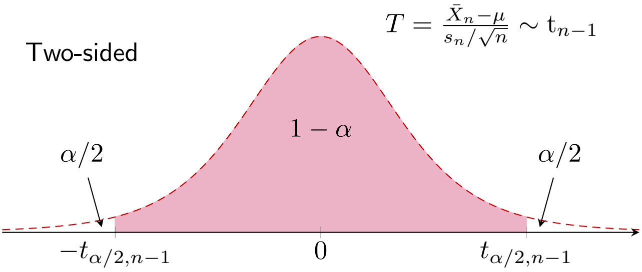

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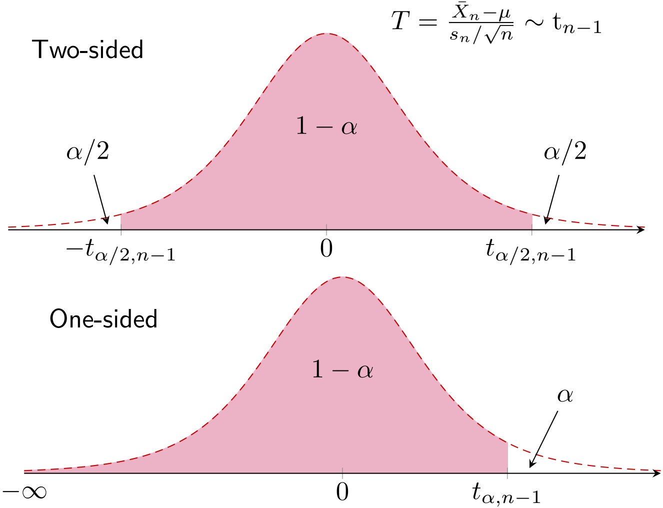
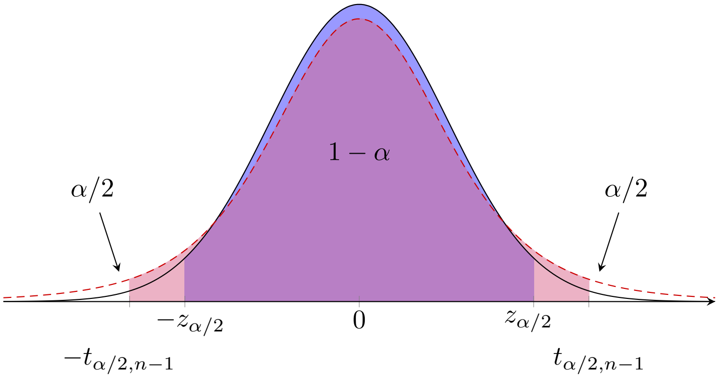


Confidence intervals for the expectation

Comparison of the critical values of N$(0,1)$ and t_{n-1}:



- Confidence intervals for μ for unknown variance σ^2 are wider than for σ^2 known.
- For $n \to +\infty$ both distributions (and thus also their critical values) coincide.

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights

Suppose that the carps' weights in a certain pond in south Bohemia are random with normal distribution $\mathrm{N}(\mu, \sigma^2)$. From 10 previously caught carps we know that:

$$\sum_{i=1}^{10} X_i = 45.65\ \text{kg} \qquad \text{and} \qquad \sum_{i=1}^{10} X_i^2 = 208.70\ \text{kg}^2.$$

Find point estimates and two-sided 90% confidence interval estimates for μ and σ^2.

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Point estimates:

- $\bar{X}_{10} = \frac{1}{10}\sum_{i=1}^{10} X_i = \frac{45.65}{10} = 4.565 \text{ kg}.$

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Point estimates:

- $\bar{X}_{10} = \frac{1}{10}\sum_{i=1}^{10} X_i = \frac{45.65}{10} = 4.565$ kg.
- $$s_{10}^2 = \frac{1}{10-1}\sum_{i=1}^{10}(X_i - \bar{X}_n)^2 = \frac{1}{10-1}\left(\sum_{i=1}^{10} X_i^2 - n(\bar{X}_n)^2\right)$$
  $$= \frac{208.7 - 10\cdot(4.565)^2}{9} = 0.0342 \text{ kg}^2.$$

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the two-sided 90% confidence interval for μ:

$$\left(\bar{X}_n - t_{\alpha/2,n-1}\frac{s_n}{\sqrt{n}}\,,\ \bar{X}_n + t_{\alpha/2,n-1}\frac{s_n}{\sqrt{n}}\right)$$

$$\left(4.565 - 1.833\frac{\sqrt{0.0342}}{\sqrt{10}}\,,\ 4.565 + 1.833\frac{\sqrt{0.0342}}{\sqrt{10}}\right)$$

$$\bar{X}_{10} = 4.565 \text{ kg}$$

$$s_{10}^2 = 0.0342 \text{ kg}^2$$

$$\alpha = 10\% = 0.1$$

$$t_{0.05,9} = 1.833$$

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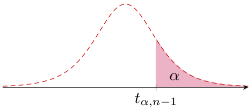
$$\alpha = 10\% = 0.1$$
$$t_{0.05,9} = 1.833$$

The **two-sided 90% confidence interval for μ** is

$$(4.4578\,,\ 4.6722) \text{ kg.}$$

Table of the critical values of the Student's t-distribution t_{n-1}



n	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	$\boxed{1.833}$	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the lower 90% confidence interval for μ:

$$\left(-\infty\,,\ \bar{X}_n + t_{\alpha,n-1}\frac{s_n}{\sqrt{n}}\right)$$

$$\left(-\infty\,,\ 4.565 + 1.383\frac{\sqrt{0.0342}}{\sqrt{10}}\right)$$

$$\bar{X}_{10} = 4.565 \text{ kg}$$

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The **lower 90% confidence interval for μ** is then

$$(-\infty\ ,\ 4.646) \text{ kg.}$$

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If the fish seller tell us that the expected weight is 4.8 kg, we can say with 90% certainty that it is not true.

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The **lower 90% confidence interval for μ** is then

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If the fish seller tell us that the expected weight is 4.8 kg, we can say with 90% certainty that it is not true.

Such considerations form the basis of *hypothesis testing* (see later).

Confidence intervals for the variance

Theorem

Suppose we observe a random sample $X_1, \ldots, X_n$ from the normal distribution $\mathrm{N}(\mu, \sigma^2)$. The two-sided $100 \cdot (1-\alpha)\%$ confidence interval for σ^2 is

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha/2,n-1}} \,,\, \frac{(n-1)s_n^2}{\chi^2_{1-\alpha/2,n-1}} \right),$$

where $\chi^2_{\alpha/2,n-1}$ is the ***critical value*** *of the χ^2 distribution with $n-1$ degrees of freedom, i.e., $\mathrm{P}(X > \chi^2_{\alpha/2,n-1}) = \alpha/2$ if $X \sim \chi^2_{n-1}$.*

The one-sided $100 \cdot (1-\alpha)\%$ confidence intervals for σ^2 are then

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha,n-1}} \,,\, +\infty \right) \quad \text{and} \quad \left(0 \,,\, \frac{(n-1)s_n^2}{\chi^2_{1-\alpha,n-1}} \right).$$

Confidence intervals for the variance

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The one-sided $100 \cdot (1-\alpha)\%$ confidence intervals for σ^2 are then

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha,n-1}} , +\infty \right) \quad \textit{and} \quad \left(0 , \frac{(n-1)s_n^2}{\chi^2_{1-\alpha,n-1}} \right).$$

✓ The statement holds only for the normal distribution!

Confidence intervals for the variance

Proof

We know that

$$\frac{(n-1)s_n^2}{\sigma^2}$$

has the chi-square distribution χ^2_{n-1}. Then the confidence interval can be established using the critical values:

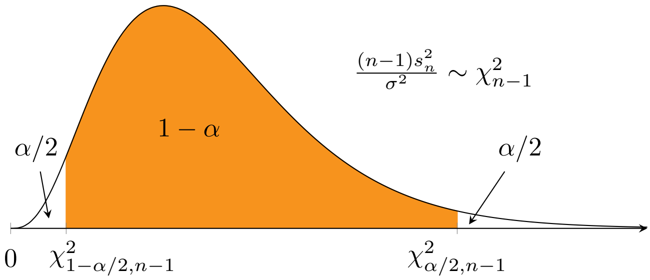
$$\mathrm{P}\left(\chi^2_{1-\alpha/2,n-1} < \frac{(n-1)s_n^2}{\sigma^2} < \chi^2_{\alpha/2,n-1}\right) = 1-\alpha.$$

By multiplying all parts by σ^2 and dividing with the critical values we get that indeed:

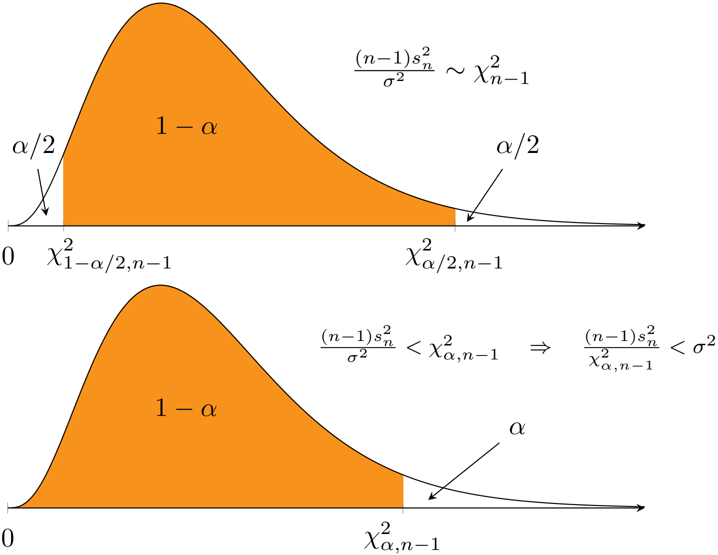
$$\mathrm{P}\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s_n^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1-\alpha.$$

$\square$

Confidence intervals for the variance



Confidence intervals for the variance



Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the two-sided 90% confidence interval for the variance σ^2 of the carps' weights:

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)s_n^2}{\chi^2_{1-\alpha/2,n-1}} \right)$$

$$\left(\frac{9 \cdot 0.0342}{16.919}, \frac{9 \cdot 0.0342}{3.325} \right)$$

$$s_{10}^2 = 0.0342\,\text{kg}^2$$

$$\alpha = 10\% = 0.1$$

$$\chi^2_{0.05,9} = 16.919$$

$$\chi^2_{0.95,9} = 3.325$$

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

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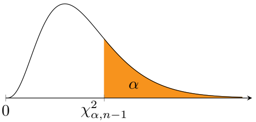
$$\chi^2_{0.95,9} = 3.325$$

The **two-sided 90% confidence interval for σ^2** is

$$(0.0182 \ , \ 0.0926) \text{ kg}^2.$$

Table of the critical values of the χ^2 distribution



n	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the upper one-sided 90% confidence interval for the variance σ^2 of the carps' weights:

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha,n-1}}\,,\ +\infty\right)$$

$$\left(\frac{9\cdot 0.0342}{14.684}\,,\ +\infty\right)$$

$$s_{10}^2 = 0.0342\ \text{kg}^2$$

$$\alpha = 10\% = 0.1$$

$$\chi^2_{0.1,9} = 14.684$$

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the upper one-sided 90% confidence interval for the variance σ^2 of the carps' weights:

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$$\left(\frac{9 \cdot 0.0342}{14.684} \, , +\infty \right)$$

$$s_{10}^2 = 0.0342 \text{ kg}^2$$

$$\alpha = 10\% = 0.1$$

$$\chi^2_{0.1,9} = 14.684$$

The **upper one-sided 90% confidence interval for σ^2** is then

$$(0.0210 \, , \, +\infty) \text{ kg}^2.$$

Estimates of μ and σ^2 – normal distribution – example

Example – fishes' weights – continuation

Find the upper one-sided 90% confidence interval for the variance σ^2 of the carps' weights:

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha,n-1}} \,,\, +\infty \right)$$

$$\left(\frac{9 \cdot 0.0342}{14.684} \,,\, +\infty \right)$$

$$s_{10}^2 = 0.0342 \text{ kg}^2$$

$$\alpha = 10\% = 0.1$$

$$\chi^2_{0.1,9} = 14.684$$

The **upper one-sided 90% confidence interval for σ^2** is then

$$(0.0210 \,,\, +\infty) \text{ kg}^2.$$

If the fish seller tell us that the variance of the weights is 0.01 kg^2, meaning that the standard deviation is 100 grams, we could say with 90% certainty that it is not true.

Recap

Confidence intervals or **interval estimates** for a parameter θ of a distribution are such bounds $L = L(\mathbf{X}), U = U(\mathbf{X})$, for which

$$\mathrm{P}(L < \theta < U) = 1 - \alpha.$$

α is chosen as small, typically 5% or 1%. Then we speak of $(1-\alpha)\%$ confidence intervals.

The **two-sided** confidence intervals for the expectation μ of a random sample from the **normal distribution** with **known variance** can be found as

$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \, , \, \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right),$$

where z denotes the corresponding critical value of the standard normal distribution.

Further cases:

- If the variance is **unknown**, use the **sample standard deviation** s_n instead of σ and critical values of the **Student's t-**distribution t_{n-1} instead of z.
- For a **one-sided** lower or upper interval, replace one bound with $\pm\infty$ and in the other bound use α instead of $\alpha/2$.
- To obtain confidence intervals for the **variance** σ^2, use the approach based on the **chi-square** distribution χ^2_{n-1}.