Hypothesis testing

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

Suppose we observe a **random sample** $X_1, \ldots, X_n$ (independent and identically distributed random variables) from an **unknown distribution**. We aim to estimate:

- The **shape** of the distribution – its type and parametric family.
- The **parameters** of the distribution.

The expectation $\mathrm{E}\, X_i = \mu$ and the variance $\mathrm{var}\, X_i = \sigma^2$ are most often estimated by the sample mean and the sample variance:

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \qquad s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2.$$

Confidence intervals or **interval estimates** for a parameter θ of a distribution are such bounds $L = L(\mathbf{X})$, $U = U(\mathbf{X})$, for which

$$\mathrm{P}(L < \theta < U) = 1 - \alpha.$$

α is chosen as small, typically 5% or 1%. Then we speak of $(1-\alpha)\%$-confidence intervals. The **two-sided** confidence intervals for the expectation μ of a random sample from the **normal distribution** with **known variance** can be found as

$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\, , \, \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right),$$

where z denotes the corresponding critical value of the standard normal distribution.

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We measure samples of n_1 and n_2 products manufactured using the old and the new procedure. On the basis of these samples we need to decide whether there is a difference between the old and the new method or not.

Hypothesis testing – hypotheses

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Statements about this distribution, which cannot be surely confirmed, are called **hypotheses**.

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Hypotheses types:

- **Non-parametric** – random sample from a general distribution. The statements deal with various properties of the distribution (e.g., median), or the shape of the distribution (goodness-of-fit tests).
- **Parametric** – random sample from a distribution given by parameters $\theta \in \mathbb{R}^d$. We test statements regarding the value of θ.

Hypothesis testing – types of errors

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- We proceed so that the probability of making the type I error is less than or equal to some small given α, which is called the **level of significance**.
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- We proceed so that the probability of making the type I error is less than or equal to some small given α, which is called the **level of significance**.
- Usually we take $\alpha = 5\%$ or $\alpha = 1\%$.
- The type II error can be either small or large depending on the sample size.
- The probability of **not making** type II error is called the **power** of the test.

Hypothesis testing – possible results

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As a null hypothesis we choose that one, where the wrongful rejection, i.e., making type I error, would be more serious.

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Rejection of H_0 in favor of H_A is a strong result.

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- If we do not reject H_0, the statement of H_A is called "statistically insignificant".

Parametric tests and confidence intervals – two-sided

Let $X_1, \ldots, X_n$ be a random sample from a distribution with a parameter θ.

We want to test the hypothesis (**two-sided alternative**):

$$H_0 : \theta = \theta_0 \qquad \text{against} \qquad H_A : \theta \neq \theta_0,$$

for some fixed value θ_0.

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Let further $\big(L(\mathbf{X}), U(\mathbf{X})\big)$ be the two-sided $100(1-\alpha)\%$ confidence interval for θ based on a random sample. Thus it holds that

$$\mathrm{P}\left(\theta \in (L, U)\right) = 1 - \alpha.$$

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If the null hypothesis H_0 holds, i.e., $\theta = \theta_0$, for the **type I error** we have

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because (L, U) is the $100(1-\alpha)\%$ confidence interval for θ.

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There are more possible decision rules. However, it can be shown (see literature) that for a general class of distributions, using the $(1-\alpha)$ confidence intervals, the probability of making type II error is lowest for any test with level of significance α. Therefore we can obtain the **most powerful test** against the given alternative.

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- If θ_0 is outside the interval, the alternative H_A holds with a large probability.

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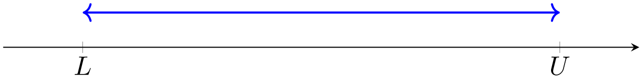
- If θ_0 is outside the interval, the alternative H_A holds with a large probability.
- The **level of significance** is again α.
- We proceed analogously for $H_A : \theta < \theta_0$.
- The null hypothesis can also be formulated in a compound form as:

$$H_0 : \theta \leq \theta_0 \qquad \text{against} \qquad H_A : \theta > \theta_0.$$

Parametric tests – illustration

Reject $H_0 : \theta = \theta_0$ in favor of the two-sided alternative $H_A : \theta \neq \theta_0$, if θ_0 does not lie in the **two-sided** confidence interval.



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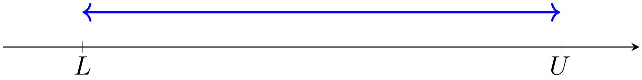

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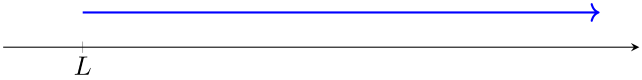

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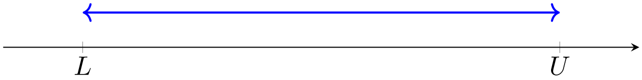

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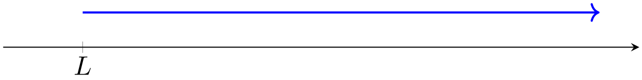

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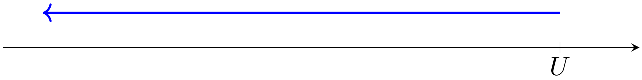

Parametric tests – testing procedure

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Testing procedure:

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- Measure (observe) the random sample.
- Construct a $(1-\alpha)$ confidence interval corresponding to the alternative hypothesis H_A.
- Reject H_0 if θ_0 is outside of the confidence interval.

Tests for the parameters of the normal distribution

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

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Let $X_1, \ldots, X_n$ be a random sample from N(μ, σ^2).

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- For known variance σ^2 we reject hypothesis H_0 if μ_0 is not in the interval

$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \, , \; \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right).$$

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$H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_A : \sigma^2 \neq \sigma_0^2$ at the level of significance α:

Tests for the parameters of the normal distribution

Let $X_1, \dots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

$H_0 : \mu = \mu_0$ against the alternative $H_A : \mu \neq \mu_0$ at the level of significance α:

- For known variance σ^2 we reject hypothesis H_0 if μ_0 is not in the interval

$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \, , \, \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right).$$

- For unknown variance σ^2 we reject hypothesis H_0 if μ_0 is not in the interval

$$\left(\bar{X}_n - t_{\alpha/2,n-1} \frac{s_n}{\sqrt{n}} \, , \, \bar{X}_n + t_{\alpha/2,n-1} \frac{s_n}{\sqrt{n}} \right).$$

$H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_A : \sigma^2 \neq \sigma_0^2$ at the level of significance α:

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$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha/2,n-1}} \, , \, \frac{(n-1)s_n^2}{\chi^2_{1-\alpha/2,n-1}} \right).$$

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Tests for parameter of normal distribution – example

Example

We have $n = 35$ observations of random variable with distribution $\mu = \mathrm{E}\,X$:

$$\text{90\% interval } A: \quad (0.4055, 5.3945)$$

$$\text{95\% interval } B: \quad (-0.0724, 5.8724)$$

Test hypothesis

$$H_0 : \mu = 0 \qquad \textit{against} \qquad H_A : \mu > 0$$

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- 2.5% – $(-0.0724, +\infty)$ and because $0 \in (-0.0724, +\infty)$ we cannot reject the null hypothesis at significance level $\alpha = 2.5\%$.

Hypothesis testing – p-value

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Meaning of the p-value

- Many statistical softwares give only the p-value as the output of a hypothesis test.
- If the p-value is smaller than our required significance level α we reject H_0.
- The size of the p-value informs us how strong is the rejection of H_0 is, or how weak the non-rejection.
- The smaller the p-value is, the more significant is the rejection of H_0.

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We know that if H_0 holds and σ^2 is known, it holds that

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Therefore with a large probability $1 - \alpha$, the standardised distance should be within bounds given by the critical values of $\mathsf{N}(0,1)$.

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Therefore with a large probability $1 - \alpha$, the standardised distance should be within bounds given by the critical values of $\mathsf{N}(0, 1)$. We can thus reject H_0 in favor of $H_A : \mu \neq \mu_0$ if

$$\left| \frac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n} \right| > z_{\alpha/2}.$$

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After separating μ_0 we obtain the same interval as the corresponding $(1 - \alpha)$ confidence interval.

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✓ The critical region can be often converted to the corresponding confidence interval.

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$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$		$T > \chi^2_{\alpha/2,n-1} \vee T < \chi_{1-\alpha/2,n-1}$
$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$T = \dfrac{(n-1)s_n^2}{\sigma_0^2}$	$T > \chi^2_{\alpha,n-1}$
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Similarly for one-sided alternatives.

Paired t-test – example

Example – comparing fathers' and sons' heights

Suppose we want to determine whether the men's height increases between generations. We have observed five pairs of fathers and their sons, now adults. Their height was measured as follows (in centimeters):

height of father	X_i	172	176	180	184	186
height of son	Y_i	178	188	177	192	193
difference	$Z_i = Y_i - X_i$	6	12	-3	8	7

We test whether the expected sons' height is equal to the expected fathers' height, against the alternative that sons are significantly taller, using $\alpha = 5\%$.

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The upper one-sided 95% confidence interval for the expectation μ_{diff} of Z_i is

$$\left(\bar{Z}_n - t_{\alpha,n-1}\frac{s_Z}{\sqrt{n}}\,,\ +\infty\right) = \left(6 - 2.132\cdot\frac{5.52}{\sqrt{5}}\,,\ +\infty\right) = (0.735, +\infty).$$

The tested value $\mu_{\text{diff}} = 0$ does not lie in the interval, so we can reject the hypothesis in favor of the alternative that the sons are significantly taller than their fathers.
The test statistic and the p-value can be obtained in R using:

```
t.test(height_son,height_father,paired=T,alternative="greater")
```

Two sample t-test

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It can be shown that if H_0 holds, the statistic

$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_\bullet}$$

follows the Student's t-distribution. The sample standard deviation $s_\bullet$ and the number of degrees of freedom depend on whether the samples have equal variances ($\sigma_1^2 = \sigma_2^2$) or not.

Two sample t-test

Suppose we observe a random sample of $X_1, \dots, X_{n_1}$ and an independent sample of $Y_1, \dots, Y_{n_2}$.

Such a situation can describe the value of a certain marker measured on two independent groups of patients, each undergoing a different treatment. We want to determine whether the marker is equal for both treatment groups, or whether it differs significantly.

Suppose that all variables are normally distributed with $X_i \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_i \sim \mathrm{N}(\mu_2, \sigma_2^2)$. We want to test $H_0 : \mu_1 = \mu_2$.

It can be shown that if H_0 holds, the statistic

$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_\bullet}$$

follows the Student's t-distribution. The sample standard deviation $s_\bullet$ and the number of degrees of freedom depend on whether the samples have equal variances ($\sigma_1^2 = \sigma_2^2$) or not.

The test can then be performed by comparing the test statistic T with the corresponding critical values of the t-distribution.

Two-sample tests – normal distribution

Let $X_1, \ldots, X_{n_1}$ be a random sample from N(μ_1, σ_1^2) and $Y_1, \ldots, Y_{n_2}$ be a random sample from N(μ_2, σ_2^2).

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Tests for the equality of expectations under $\sigma_1^2 = \sigma_2^2$:

H_0	H_A	test statistic T	critical region W_α
$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$	$T = \dfrac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_{12}} \sqrt{\dfrac{n_1 n_2}{n_1 + n_2}}$	$\lvert T \rvert > t_{\alpha/2, n_1+n_2-2}$
$\mu_1 \leq \mu_2$	$\mu_1 > \mu_2$		$T > t_{\alpha, n_1+n_2-2}$
$\mu_1 \geq \mu_2$	$\mu_1 < \mu_2$		$T < -t_{\alpha, n_1+n_2-2}$

- Where $s_{12} = \sqrt{\dfrac{(n_1-1)s_X^2 + (n_2-1)s_Y^2}{n_1+n_2-2}}$,
- t_{α, n_1+n_2-2} is the critical value of Student's t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Two-sample tests – normal distribution

Let $X_1, \ldots, X_{n_1}$ be a random sample from N(μ_1, σ_1^2) and $Y_1, \ldots, Y_{n_2}$ be a random sample from N(μ_2, σ_2^2).

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Tests for the equality of expectations under $\sigma_1^2 \neq \sigma_2^2$:

H_0	H_A	test statistic T	critical region W_α
$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$	$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_d}$	$\lvert T\rvert > t_{\alpha/2, n_d}$
$\mu_1 \leq \mu_2$	$\mu_1 > \mu_2$		$T > t_{\alpha, n_d}$
$\mu_1 \geq \mu_2$	$\mu_1 < \mu_2$		$T < -t_{\alpha, n_d}$

- Where $s_d = \sqrt{\frac{s_X^2}{n_1} + \frac{s_Y^2}{n_2}}$,
- $n_d = \dfrac{s_d^4}{\frac{1}{n_1 - 1}\left(\frac{s_X^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_Y^2}{n_2}\right)^2}$

Two sample t-test – example

Example – comparing men's heights from different countries

Suppose we want to determine whether the average men's height is the same in the Czech Republic and in Norway. We have observed five men from CZE and six men from NOR. Their heights were measured as follows (in centimeters):

height CZE	X_i	169	178	179	186	191	
height NOR	Y_i	175	182	183	189	191	192

We test whether the expected heights are equal, against the alternative that they are not, on $\alpha = 5\%$. We take the variances as equal.

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$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_{12}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}} = -1.0545.$$

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Since

$$1.0545 = |T| < t_{\alpha/2, n_1+n_2-2} = 2.262,$$

we do not reject the null hypothesis of equality. Based on our data we could not find a significant difference between the expected heights of men among the two countries.

Two sample t-test – example

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The test statistic and the p-value can be obtained in R using:

```
t.test(height_cze,height_nor,paired=F,alternative="two.sided")
```

Two-sample tests – normal distribution

Let $X_1, \dots, X_{n_1}$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_1, \dots, Y_{n_2}$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$.

Two-sample tests – normal distribution

Let $X_1, \ldots, X_{n_1}$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2}$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$.

Tests for the equality of variances – F-test:

H_0	H_A	test statistic T	critical region W_α
$\sigma_1^2 = \sigma_2^2$	$\sigma_1^2 \neq \sigma_2^2$		$T < F_{1-\alpha/2,n_1-1,n_2-1} \vee T > F_{\alpha/2,n_1-1,n_2-1}$
$\sigma_1^2 \leq \sigma_2^2$	$\sigma_1^2 > \sigma_2^2$	$T = \frac{s_X^2}{s_Y^2}$	$T > F_{\alpha,n_1-1,n_2-1}$
$\sigma_1^2 \geq \sigma_2^2$	$\sigma_1^2 < \sigma_2^2$		$T < F_{1-\alpha,n_1-1,n_2-1}$

- s_X^2 is the sample variance of the first random sample and s_Y^2 is the sample variance of the second sample.
- F_{α,n_1-1,n_2-1} is the critical value of the *Fisher-Snedecor F-distribution* with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.
- Important note: The F-test is particularly sensitive to the normality of X and Y. If we are not sure whether the data is normally distributed, it is better to use a different test or assume non-equal variances for the t-test.
- The test can be called in R using `var.test(height_cze,height_nor)`.

Recap

We want to test a **hypothesis** concerning a parameter of a distribution.

We test the **null hypothesis** H_0 against an **alternative hypothesis** H_A.

We set up a decision rule, which, based on the observed data, either **rejects** or **does not reject** H_0 in favor of H_A.

We don't want the probability of falsely rejecting H_0 to exceed a chosen **level of significance** α.

We can establish the decision rule based on the confidence intervals:

- Reject $H_0 : \theta = \theta_0$ in favor of $H_A : \theta \neq \theta_0$, if θ_0 does not lie in the $1 - \alpha$ two-sided confidence interval.
- Reject $H_0 : \theta = \theta_0$ in favor of $H_A : \theta > \theta_0$, if θ_0 does not lie in the $1 - \alpha$ one-sided upper confidence interval.
- Reject $H_0 : \theta = \theta_0$ in favor of $H_A : \theta < \theta_0$, if θ_0 does not lie in the $1 - \alpha$ one-sided lower confidence interval.

Based on the **test statistics** approach we can also solve paired and two-sample problems.