Conditional probability and independence

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, covariance and correlation, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

A random experiment is represented using a probability space $(\Omega, \mathcal{F}, \mathrm{P})$:

- Ω is the set of possible results;
- $\mathcal{F}$ is a system of subsets of Ω;
- elements $A \in \mathcal{F}$ are called random events;
- the probability measure P is a function, which assigns to the random events a real value from 0 to 1, representing the ideal proportion of cases, in which the events occur.

If there is only a finite many possible results with equal probabilities, then

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|}.$$

Conditional probability

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If we know that an even number was rolled, then it is clear that $\mathrm{P}(4 \mid \text{even}) = 1/3$.

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If we know that an event B surely occurred, we are in fact interested only in outcomes of the experiment favorable to the event B. Favorable outcomes to the event A are now in $A \cap B$ and all of them must be in B (B surely occurred). We have

$$\mathrm{P}(A|B) = \frac{\text{size}(A \cap B)}{\text{size}(B)} = \frac{\text{size}(A \cap B)/\text{size}(\Omega)}{\text{size}(B)/\text{size}(\Omega)} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

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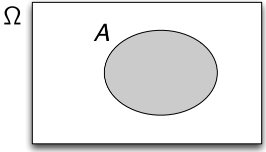
Definition

Let A, B be events and $\mathrm{P}(B) > 0$. The **conditional probability** of the event A given (the event) B is denoted by $\mathrm{P}(A|B)$ and is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Conditional probability – Venn diagram



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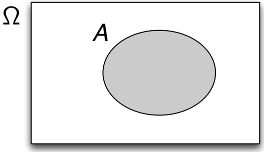

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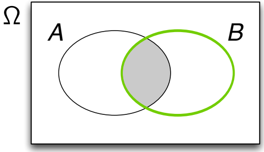

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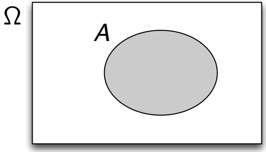

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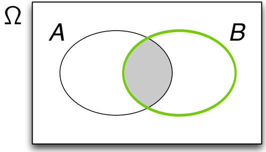

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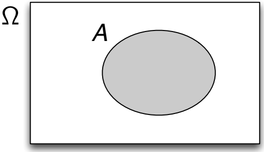

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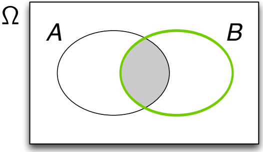

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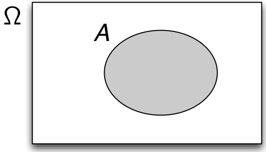

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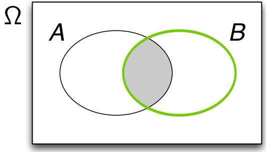

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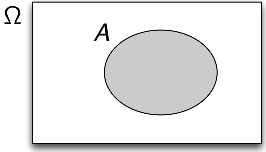

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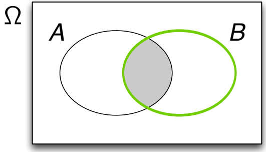

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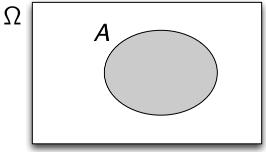

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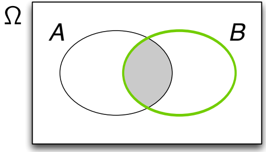

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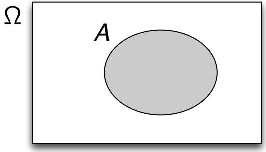

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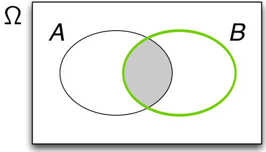

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Formally: $\Omega = \{1, 2, 3, 4, 5, 6\}^2$,
$\mathrm{P}(A) = |A|/36$ for each $A \subset \Omega$.
Let $B = \{(3, \omega_2) : 1 \leq \omega_2 \leq 6\}, \;\; A = \{(\omega_1, \omega_2) : \omega_1 + \omega_2 > 6\}$.

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Then

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\frac{|A \cap B|}{36}}{\frac{|B|}{36}} = \frac{|A \cap B|}{|B|} = \frac{3}{6}.$$

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Example – family with two children

A trickier example:

A family has two children. What is the probability that both are boys, given that at least one of them is a boy? I.e., what is the value of $\mathrm{P}(\text{both boys} \mid \text{at least one is a boy})$?

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Incorrect: $\mathrm{P}(BB|\text{older is boy}) = \mathrm{P}(BB|BG \cup BB) = \dfrac{\mathrm{P}(BB \cap (BG \cup BB))}{\mathrm{P}(BG \cup BB)} = \dfrac{1}{2}.$

Properties of conditional probability

Lemma

Let $\mathrm{P}(B) > 0$. *Then the conditional probability* $\mathrm{P}(\cdot|B)$ *is a probability measure, i.e.,* $\mathrm{P}(\cdot|B) \in [0,1]$ *and it fulfills the axioms of probability.*

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- **iv)** $\sigma-$**additivity**: If $A_1, A_2, \ldots \in \mathcal{F}$ are mutually disjoint events (i.e., $A_i \cap A_j = \emptyset$ for $\forall\, i,j : i \neq j$), then

$$\mathrm{P}\left(\bigcup_{i=1}^{+\infty} A_i \middle| B\right) = \frac{\mathrm{P}\left(\left(\bigcup_{i=1}^{+\infty} A_i\right) \cap B\right)}{\mathrm{P}(B)} = \frac{\mathrm{P}\left(\bigcup_{i=1}^{+\infty} (A_i \cap B)\right)}{\mathrm{P}(B)} = \cdots = \sum_{i=1}^{+\infty} \mathrm{P}(A_i|B).$$

□

Properties of conditional probability

Conditional probability fulfills all mentioned properties of probability as well:

- if A_1 and A_2 are mutually disjoint, then $\mathrm{P}(A_1 \cup A_2 | B) = \mathrm{P}(A_1 | B) + \mathrm{P}(A_2 | B)$,
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Moreover, the probability $\mathrm{P}(A|B)$ “lives” on B: for $A \cap B = \emptyset$ we have $\mathrm{P}(A|B) = 0$.

Furthermore, $\mathrm{P}(A \cap B|B) = \dfrac{\mathrm{P}(A \cap B \cap B)}{\mathrm{P}(B)} = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A|B)$.

Case distinct formula (Law of total probability)

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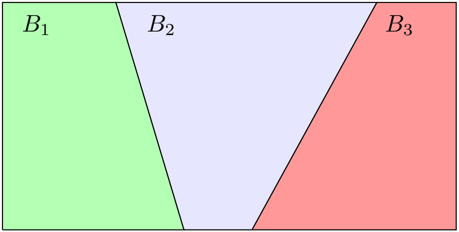

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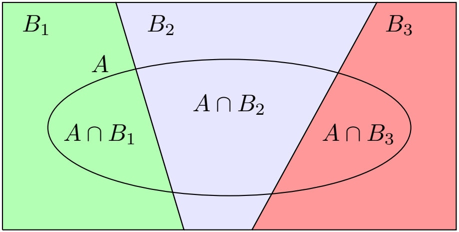

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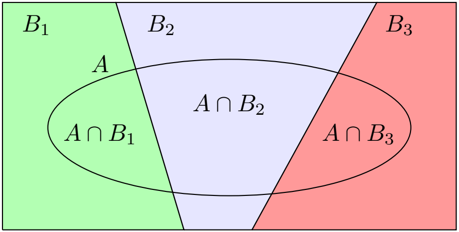

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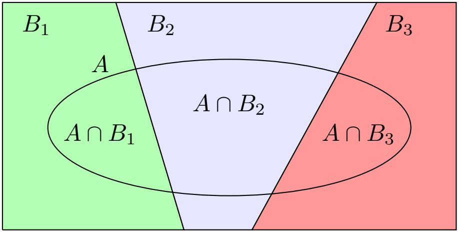

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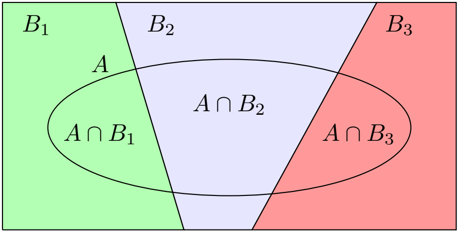

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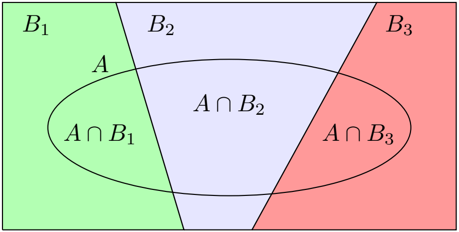

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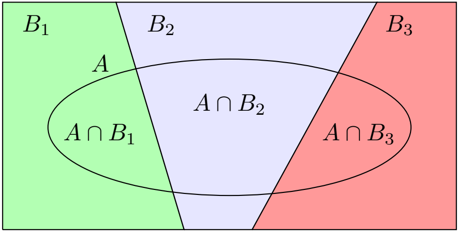

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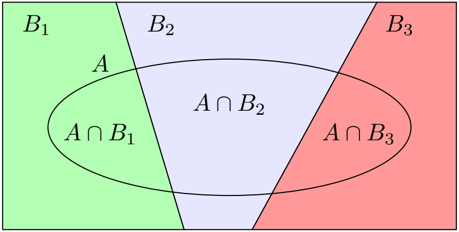

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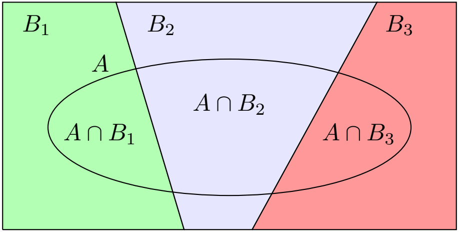

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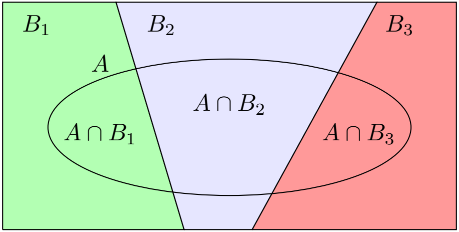

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Bayes' Theorem – example

Example – spam filter

From the analysis of our email account we find out that:

- 30% of all delivered messages is spam;
- in 70% of spam messages there is the word "copy";
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Calculate the probability that a message containing the word "copy" is a spam,

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$$\mathrm{P}(S) = 0.3,\ \mathrm{P}(S^c) = 0.7, \qquad \mathrm{P}(C|S) = 0.7,\ \mathrm{P}(C|S^c) = 0.1$$

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Probability trees

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which can be proven by using the definition of conditional probability on the right hand side:

$$\begin{aligned}\mathrm{P}(A)\,\mathrm{P}(B|A)\,\mathrm{P}(C|A \cap B) &= \mathrm{P}(A)\frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}\frac{\mathrm{P}(C \cap (A \cap B))}{\mathrm{P}(A \cap B)} \\ &= \mathrm{P}(A \cap B \cap C).\end{aligned}$$

Probability trees

Lemma – Multiplicative law

Let for events $A_1, \ldots, A_n$ *hold that* $\mathrm{P}(A_1 \cap \cdots \cap A_n) > 0$. *Then it holds that*

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Proof

We apply successively the relation $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B|A)$ following from the definition of conditional probability:

$$\begin{aligned}
\mathrm{P}(A_1 \cap \cdots \cap A_n) &= \mathrm{P}(A_1 \cap \cdots \cap A_{n-1})\,\mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1}) \\
&= \mathrm{P}(A_1 \cap \cdots \cap A_{n-2})\,\mathrm{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2})\,\mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1}) \\
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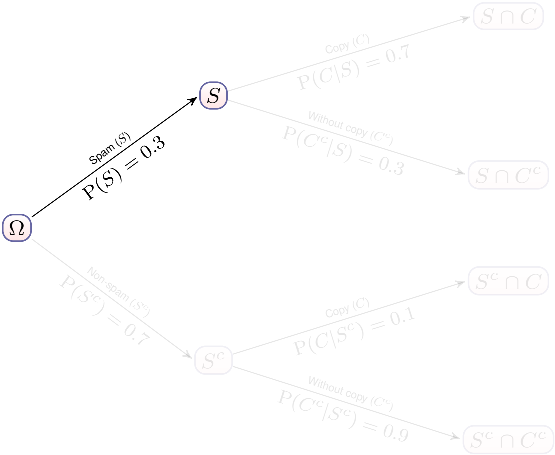
\end{aligned}$$

□

Example – spam filter



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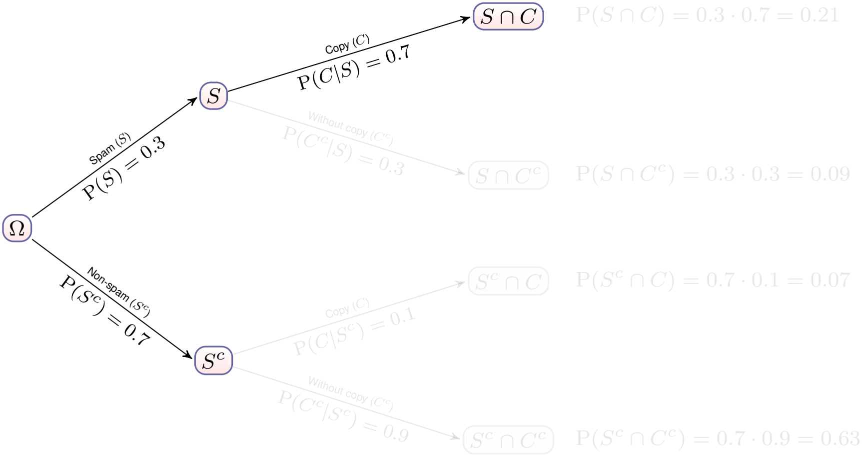

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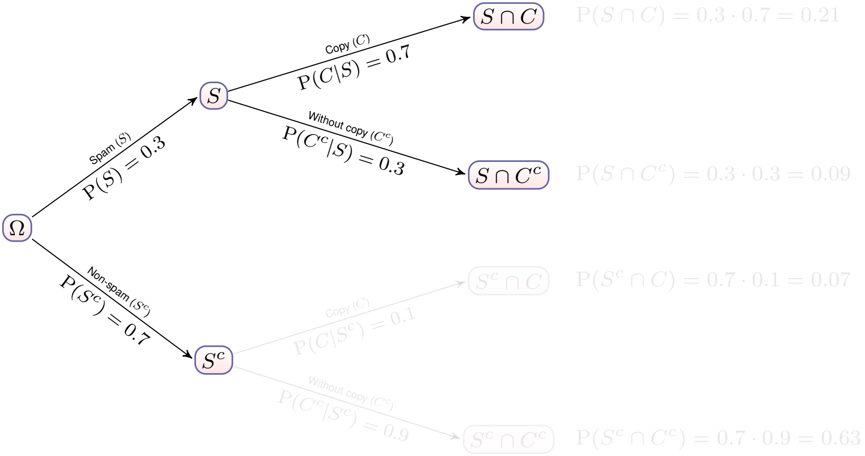

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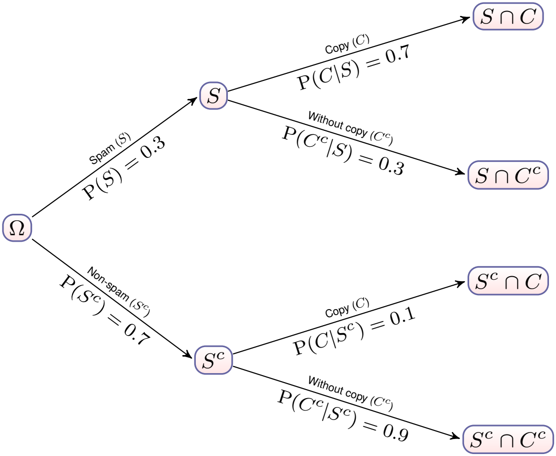

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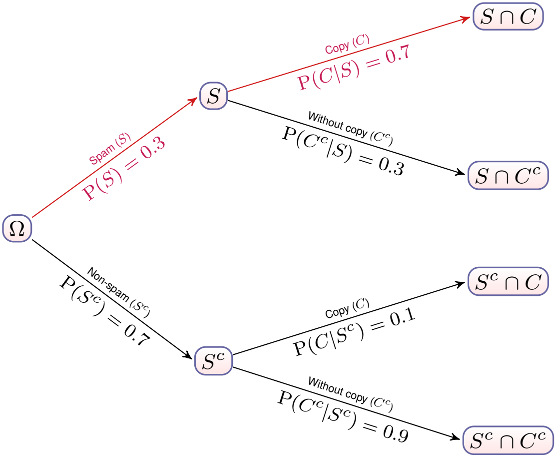

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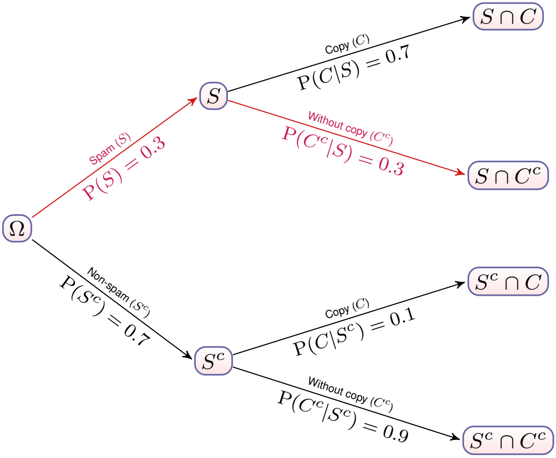

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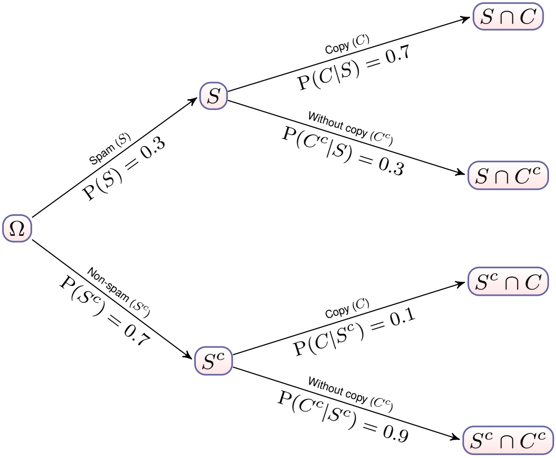

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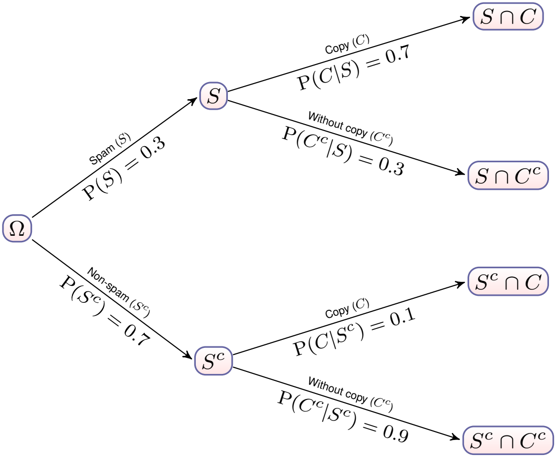

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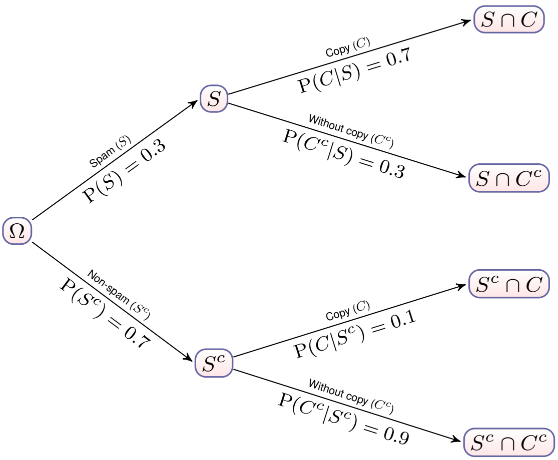

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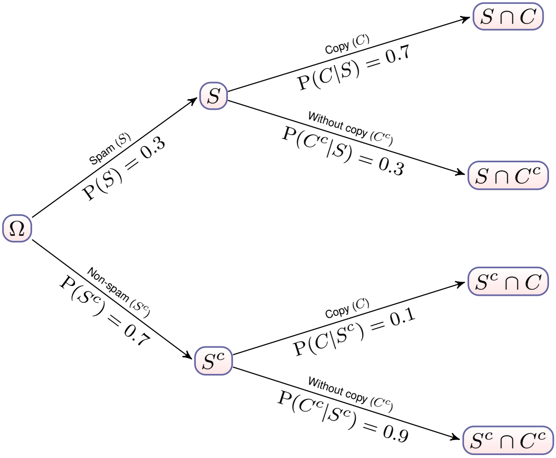

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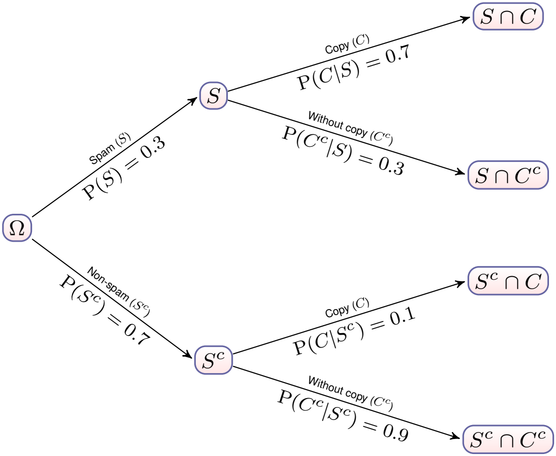

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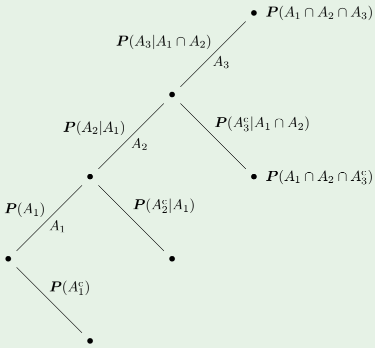

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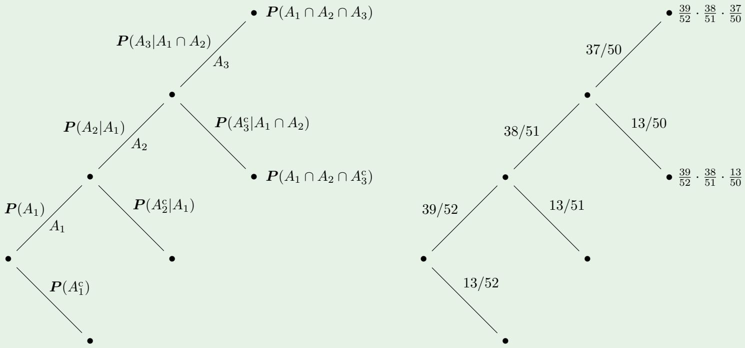

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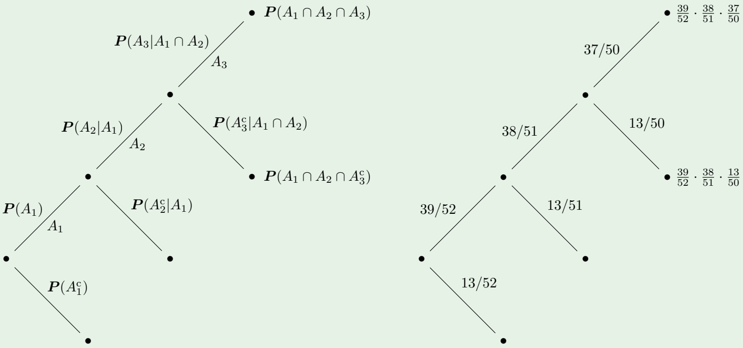

The probability of a given vertex of the tree is the product of the corresponding values on the path stemming from the root.

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$$\frac{\mathrm{P}(\text{accident}|\text{drunk})}{\mathrm{P}(\text{accident}|\text{sober})} = \frac{\mathrm{P}(\text{accident} \cap \text{drunk}) / \mathrm{P}(\text{drunk})}{\mathrm{P}(\text{accident} \cap \text{sober}) / \mathrm{P}(\text{sober})}$$

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Drunk drivers have at least 11 times higher probability of causing a fatal accident.

Independence of events

Intuitively: A and B are independent if the probability of the event A is not influenced by the knowledge about occurrence of the event B, i.e., $\mathrm{P}(A|B) = \mathrm{P}(A)$, and (vice versa) $\mathrm{P}(B|A) = \mathrm{P}(B)$.

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Generally, a family of events $\{A_i \mid i \in I\}$ is called **independent** if

$$\mathrm{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathrm{P}(A_i)$$

for all finite non-empty subsets J of I.

Independence of events

Example – rolling a die

Consider the events

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$$\mathrm{P}(A \cap B) = \frac{1}{6} \qquad \text{and} \qquad \mathrm{P}(A)\,\mathrm{P}(B) = \frac{3}{6} \cdot \frac{1}{6} = \frac{1}{12}.$$

Then events A and B are <u>not</u> independent.

Relation between independence and conditional probability

Let A and B be independent events and $\mathrm{P}(B) > 0$. Then clearly

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A)\,\mathrm{P}(B)}{\mathrm{P}(B)} = \mathrm{P}(A).$$

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If the events A and B are independent then A and B^c (resp., A^c and B; A^c and B^c) are independent, too.

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If $(A_i)_{i \in I}$ is a family of independent events, then for any arbitrary non-empty finite subset $\emptyset \neq J \subset I$ it holds that

$$\mathrm{P}\left(\bigcap_{i \in J} A_i \mid \bigcap_{i \in I \setminus J} A_i\right) = \mathrm{P}\left(\bigcap_{i \in J} A_i\right).$$

Independent vs disjoint events

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The events being disjoint is a matter of sets, independence is a matter of probabilities.

Conditional independence

Definition

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and C an event with $\mathrm{P}(C) > 0$. Events A and B are called **conditionally independent with respect to C**, if

$$\mathrm{P}(A \cap B | C) = \mathrm{P}(A|C)\,\mathrm{P}(B|C).$$

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Recall:

- $Q(A) = \mathrm{P}(A|C)$ is a probability measure;
- the conditional independence is thus the independence with respect to probability Q.

Conditional independence

Example – rolling a seven-sided die

Suppose we roll a seven-sided die with all sides equally likely. Consider the events:

A: "an even number is rolled", B: "a number less than 3 is rolled".

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Are the events A and B independent? $A = \{2,4,6\}$, $B = \{1,2\}$, $A \cap B = \{2\}$.

$$\mathrm{P}(A \cap B) = \frac{1}{7} \quad \text{and} \quad \mathrm{P}(A) \cdot \mathrm{P}(B) = \frac{3}{7} \cdot \frac{2}{7} = \frac{6}{49}.$$

Events A and B are not independent.

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Example – rolling a seven-sided die + condition

Consider further event C: "we rolled at most 6" $C = \{1,2,3,4,5,6\}$.

Are events A and B conditionally independent with respect to C?

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Consider further event C: "we rolled at most 6" $C = \{1,2,3,4,5,6\}$.

Are events A and B conditionally independent with respect to C?

$$\mathrm{P}(A \cap B|C) = \frac{\mathrm{P}(A \cap B \cap C)}{\mathrm{P}(C)} = \frac{\mathrm{P}(\{2\})}{\mathrm{P}(\{1,\dots,6\})} = \frac{1/7}{6/7} = \frac{1}{6},$$

$$\mathrm{P}(A|C) \cdot \mathrm{P}(B|C) = \frac{3/7}{6/7} \cdot \frac{2/7}{6/7} = \frac{1}{6}.$$

Events A and B are conditionally independent with respect to C.

Recap

- The **conditional probability** that an event A occurs if we know that an event B with $\mathrm{P}(B) > 0$ occured, is defined as $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.
- **Law of total probability**: For A and B with $\mathrm{P}(B) > 0$ we have

$$\mathrm{P}(A) = \mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B^c)\,\mathrm{P}(B^c).$$

- **Bayes' Theorem**: For A and B with $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$ we have

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A|B)\,\mathrm{P}(B)}{\mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B^c)\,\mathrm{P}(B^c)}.$$

Recap

- The **conditional probability** that an event A occurs if we know that an event B with $\mathrm{P}(B) > 0$ occured, is defined as $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.
- **Law of total probability**: For A and B with $\mathrm{P}(B) > 0$ we have
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$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A|B)\,\mathrm{P}(B)}{\mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B^c)\,\mathrm{P}(B^c)}.$$
- Events A and B are called **independent** if
$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B).$$
- For **independent** events A and B the knowledge that one of them occurred or not does not change the probability of the other one happening:
$$\mathrm{P}(A|B) = \mathrm{P}(A) \qquad \text{and} \qquad \mathrm{P}(B|A) = \mathrm{P}(B).$$