Random vectors I.

(Random vectors, independence, conditional distribution)

Lecturer:
Francesco Dolce

Department of Applied Mathematics
Faculty of Information Technology
Czech Technical University in Prague

Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

- A **random variable** X is a measurable function, which assigns real values to the outcomes of a random experiment.
- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

$$F_X(x) = \mathrm{P}(X \leq x).$$

- There are two major types of random variables:
  - **Discrete**, taking only countably many possible values.
  - **Continuous**, taking uncountably many values from an interval.
- The distribution can be given by:
  - for discrete distributions by the **probabilities** of possible values $\mathrm{P}(X = x_k)$.
  - for continuous distributions by the **density** f_X for which

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t.$$

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The individual variables can have different distributions and the values of the variables can be strongly mutually interconnected. It is appropriate to describe their distribution together as the so called **joint distribution**.

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Consider two random variables X and Y defined on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We define their **joint distribution function** $F_{X,Y}(x, y)$ as

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For n random variables $X_1, X_2, \ldots, X_n \overset{\text{denote}}{=} \boldsymbol{X}$ we define the joint distribution function as

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The couple (X, Y) or, n-tuple $(X_1, \ldots, X_n)$, is called a **random vector**.

Example – joint distribution

Example

Let X and Y be random variables with a joint discrete distribution given by the following probabilities:

$P(X = x \cap Y = y)$		x = 0.5	x = 1	x = 2
y	2	0.3	0.06	0.04
y	1	0.4	0.15	0.05

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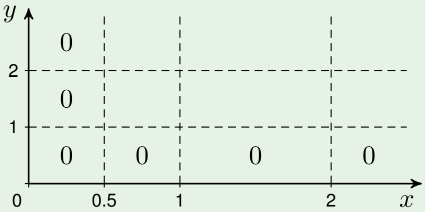

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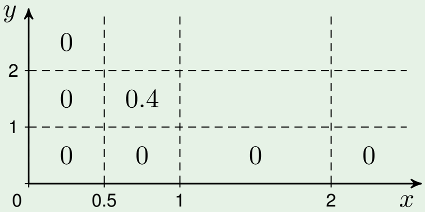

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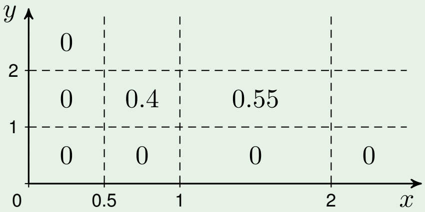

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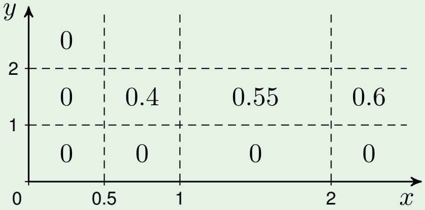

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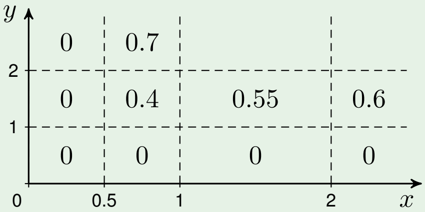

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Properties of the joint distribution function

The joint distribution function has analogous properties as the distribution function of one variable.

Theorem

The joint distribution function $F_{X,Y}$ of random variables X and Y has following properties:

i) *if $x_1 < x_2$ and $y_1 < y_2$ then $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$.*

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iii) $\forall\, y \in \mathbb{R}, \ \lim_{x \to +\infty} F_{X,Y}(x, y) = F_Y(y)$ *and*
$\forall x \in \mathbb{R}, \ \lim_{y \to +\infty} F_{X,Y}(x, y) = F_X(x).$

Proof

Analogously as for the distribution function of one random variable. □

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If the variables X and Y are discrete, it is often useful to describe the distribution by the **joint probabilities** of their values.

Definition

The joint probabilities of values of two discrete random variables X and Y is

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(\{X = x\} \cap \{Y = y\}).$$

Taken as a function of x and y, the probabilities are called the **joint probability mass function**.

Joint probabilities and the joint distribution function

The **joint distribution function** of two discrete random variables X and Y is

$$F_{X,Y}(x,y) = \mathrm{P}(X \leq x \cap Y \leq y) = \sum_{\{i:\, x_i \leq x\}} \sum_{\{j:\, y_j \leq y\}} \mathrm{P}(X = x_i \cap Y = y_j)$$

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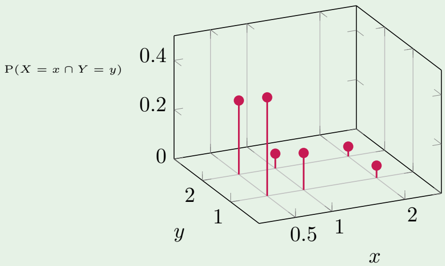
The **normalization condition** follows from the properties of the joint distribution function:

$$\sum_i \sum_j \mathrm{P}(X = x_i \cap Y = y_j) = \sum_i \mathrm{P}\left(\{X = x_i\} \cap \bigcup_j \{Y = y_j\}\right)$$
$$= \sum_i \mathrm{P}\left(\{X = x_i\} \cap \{Y \in \mathbb{R}\}\right) = \sum_i \mathrm{P}\left(X = x_i\right)$$
$$= \mathrm{P}\left(\bigcup_j \{X = x_i\}\right) = \mathrm{P}\left(\{X \in \mathbb{R}\}\right) = \mathrm{P}(\Omega) = 1.$$

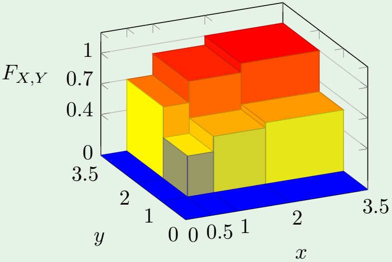
Joint distribution – visualization

Example

Joint probabilities



Joint distribution function



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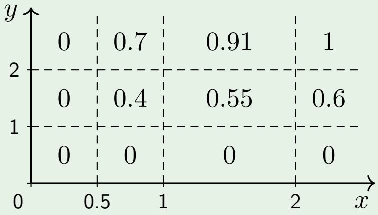

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Random variables X and Y are called independent if for all $x,\ y \in \mathbb{R}$ the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent. Equivalently, if it holds that for all $x, y \in \mathbb{R}$

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Random variables forming a countable collection $X_1, X_2, \ldots$ are called independent if all finite n-tuples $X_{i_1}, \ldots, X_{i_n}$ are independent.

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Proof

If the condition regarding equalities holds, it must hold also for all inequalities, because they can be rewritten as sums of probabilities of disjoint events.
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Example – checking independence of random variables

Example – continuation

Random variables X and Y have the following joint and marginal distributions:

$P(X = x \cap Y = y)$		x: 0.5	1	2	$P(Y = y)$
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Are X and Y independent?

No, they are not independent because, e.g., for $x = 0.5$ and $y = 2$ it holds that

$$0.3 = \mathrm{P}(X = 0.5 \cap Y = 2) \neq \mathrm{P}(X = 0.5) \cdot \mathrm{P}(Y = 2) = 0.7 \cdot 0.4 = 0.28.$$

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If the variables X and Y are continuous, it is often useful to describe the distribution by the

joint probability density.

Definition

Two random variables X and Y have a **joint (absolutely) continuous** distribution if there exists a **non-negative** function $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that for all $x, y \in \mathbb{R}$ it holds

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v) \,\mathrm{d}u\,\mathrm{d}v.$$

The function $f_{X,Y}$ is called the **joint probability density** of the random variables X, Y or of the random vector (X, Y).

Properties of continuous random variables

Similarly as in the one-dimensional case it holds that:

- Where the derivative exists:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y).$$

Properties of continuous random variables

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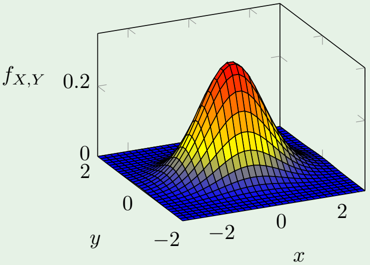
- For all B Borel subset of $\mathbb{R}^2$ (meaning that $\{X \in B\}$ is an event)

$$\mathrm{P}\left((X,Y) \in B\right) = \iint_B f_{X,Y}(x,y) \; \mathrm{d}x \; \mathrm{d}y.$$

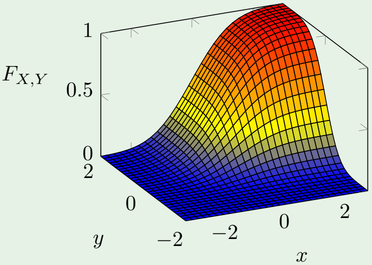
Joint distribution – visualization

Example

Joint density



Joint distribution function



$$f_{X,Y}(x,y) = \frac{1}{\pi} e^{-\frac{x^2}{2} - 2y^2}$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} \frac{1}{\pi} e^{-\frac{u^2}{2} - 2v^2} \, \mathrm{d}u \, \mathrm{d}v$$

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*Let X and Y be two random variables having a joint continuous distribution with joint density $f_{X,Y}$. Then X and Y are both continuous too, and the **marginal densities** f_X, f_Y are given by*

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y, \qquad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}x.$$

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Proof

We know that:

$$F_X(x) = \mathrm{P}(X \leq x) = \mathrm{P}(X \leq x \cap Y \in \mathbb{R}) = \int_{-\infty}^{x} \left(\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,\mathrm{d}v \right) \mathrm{d}u\,.$$

The statement of the theorem is obtained by differentiating with respect to x, or by comparing this formula to the definition of the distribution function of a continuous random variable. The second part is analogous. □

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Random variables $X_1, \ldots, X_n$ are called independent if for all $\boldsymbol{x} \in \mathbb{R}^n$

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Proof

Two random variables X and Y are independent if

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y).$$

Taking the derivatives of both sides with respect to both x and y yields one implication. Integrating both sides of the equality for densities yields the other direction. □

Independence of continuous random variables

Remark

While verifying the independence of X and Y we can use the following.

Consequence: If it is possible to decompose $f_{X,Y}$ to

$$f_{X,Y}(x,y) = g(x) \cdot h(y), \quad \forall x, y \in \mathbb{R},$$

where $g(x)$ and $h(y)$ are non-negative functions, then the variables X and Y are independent.

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✓ Do the proof yourself by inserting into the formula for marginal densities.

✓ Beware, the functions $g(x)$ and $h(y)$ may not necessarily be the marginal densities $f_X(x)$ and $f_Y(y)$; they may differ by a multiplicative constant.

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The statement of the consequence can be formulated for independence of a general random vector $X_1, \ldots, X_n$ too.

Example – marginal distribution and independence

Example

Let X and Y random variables having the joint probability density

$$f_{X,Y}(x,y) = ye^{-2x} \qquad \text{for } x \in [0,+\infty) \text{ and } y \in [0,2].$$

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Independence:

$$ye^{-2x} = f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = 2e^{-2x}\cdot\frac{y}{2} = ye^{-2x}.$$

Yes, they are independent!

Discrete conditional distribution

Now we will study the distribution of a random variable X under the assumption that we know the value of the variable $Y = y$.

Suppose that we have a partial information about the result of an experiment and we are interested in the change in our prediction.

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Definition

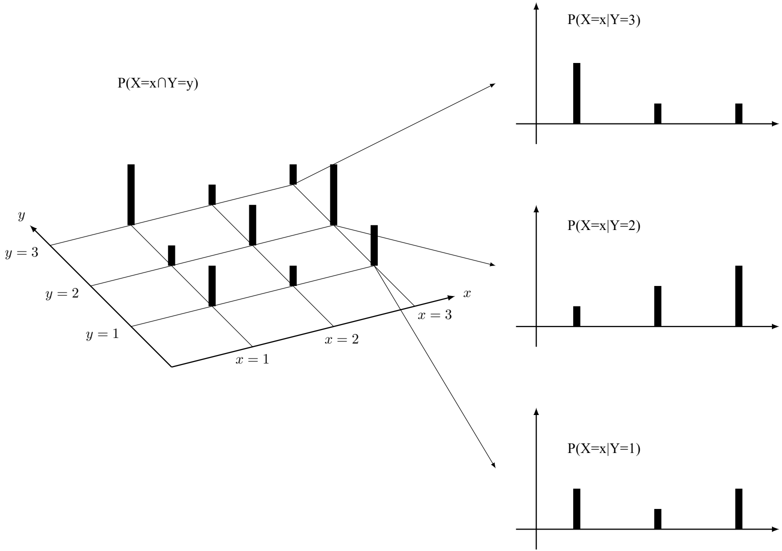
Let $\mathrm{P}(Y = y) > 0$. Then, the **conditional distribution function** $F_{X|Y}(\cdot|y)$ of the variable X given $Y = y$ is defined as

$$F_{X|Y}(x|y) = \mathrm{P}(X \leq x|Y = y).$$

The **conditional probabilities of values** of X given (under the condition of) $Y = y$ are given, analogously, by

$$\mathrm{P}(X = x|Y = y).$$

Illustration of conditional probabilities $\mathrm{P}(X = x|Y = y)$



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Thus it holds that:

$$\mathrm{E}(X|Y = y) = \sum_i x_i \, \mathrm{P}(X = x_i | Y = y) = \sum_i x_i \frac{\mathrm{P}(X = x_i \cap Y = y)}{\mathrm{P}(Y = y)}.$$

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After taking a limit $\Delta y \to 0$ we intuitively obtain the result as

$$\mathrm{P}(X \leq x | Y = y) = \int_{-\infty}^{x} \frac{f_{X,Y}(u, y)}{f_Y(y)}\,\mathrm{d}u.$$

Continuous conditional distribution

the previous inference lead us to the following formal definition:

Definition

The **conditional distribution function** of a variable X given (under the condition of) $Y = y$ is defined as

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} \frac{f_{X,Y}(u,y)}{f_Y(y)} \mathrm{d}u,$$

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The conditional density is defined accordingly:

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Analogously as in the discrete case we define the **conditional expectation** for continuous random variables:

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We compute the conditional expectation for a given value y as follows:

$$\mathrm{E}(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,\mathrm{d}x = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)}\,\mathrm{d}x = g(y),$$

where g is a function which arises from the integration.

Bayes' Theorem

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If X is discrete random variable than we have:

$$\mathrm{P}(X = n|Y = y) = \frac{\mathrm{P}(X = x)\, f_{Y|X}(y|x)}{\sum_k \mathrm{P}(X = k)\, f_{Y|X}(y, k)}.$$

Recap

Joint distribution function of a random vector (X, Y):

$$F_{X,Y}(x, y) = \mathrm{P}(X \le x \cap Y \le y).$$

Discrete random variables X and Y

Joint probabilities of values:

$$\mathrm{P}(X = x \cap Y = y)$$

Continuous random variables X and Y

Joint density:

$$f_{X,Y}(x, y)$$

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$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$$

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Independence of X and Y:

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y)$$

$$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$$

Conditional probabilities / density of X given $Y = y$:

$$\mathrm{P}(X = x | Y = y) = \frac{\mathrm{P}(X = x \cap Y = y)}{\mathrm{P}(Y = y)}$$

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Conditional expectation of X given $Y = y$:

$$\mathrm{E}(X | Y = y) = \sum_{x} x\, \mathrm{P}(X = x | Y = y)$$

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