Random vectors II.

(Covariance and correlation, convolution)

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

Joint distribution function of a random vector (X, Y):

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x \cap Y \leq y).$$

Discrete random variables X and Y	**Continuous random variables X and Y**
Joint probabilities of values: $\mathrm{P}(X = x \cap Y = y)$	Joint density: $f_{X,Y}(x, y)$

Marginal distributions:

$$\mathrm{P}(X = x) = \sum_{\text{all } y} \mathrm{P}(X = x \cap Y = y) \qquad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$$

$$\mathrm{P}(Y = y) = \sum_{\text{all } x} \mathrm{P}(X = x \cap Y = y) \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x$$

Independence of X and Y:

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y) \qquad f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$$

Conditional probabilities / density of X given $Y = y$:

$$\mathrm{P}(X = x | Y = y) = \frac{\mathrm{P}(X = x \cap Y = y)}{\mathrm{P}(Y = y)} \qquad f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Conditional expectation of X given $Y = y$:

$$\mathrm{E}(X | Y = y) = \sum_x x\,\mathrm{P}(X = x | Y = y) \qquad \mathrm{E}(X | Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,\mathrm{d}x$$

Functions of random vectors of variables

Similar formulas as for a function of one random variable also hold for the functions of random vectors. Let

$$Z = h(X_1, \dots, X_n) = h(\boldsymbol{X}).$$

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- When variables $X_1, \ldots, X_n$ have a joint discrete distribution with probabilities $\mathrm{P}(\boldsymbol{X} = \boldsymbol{x})$, the following relation holds for the distribution function of Z:

$$F_Z(z) = \mathrm{P}(Z \leq z) = \sum_{\{\boldsymbol{x} \in \mathbb{R}^n :\, h(\boldsymbol{x}) \leq z\}} \mathrm{P}(\boldsymbol{X} = \boldsymbol{x}).$$

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- When variables $X_1, \dots, X_n$ have a joint continuous distribution with density $f_{\boldsymbol{X}}(\boldsymbol{x})$, the distribution function of Z is then

$$F_Z(z) = \mathrm{P}(Z \leq z) = \int \cdots \int_{\{\boldsymbol{x} \in \mathbb{R}^n : \, h(\boldsymbol{x}) \leq z\}} f_{\boldsymbol{X}}(\boldsymbol{x}) \, \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Expected value of the function of a random vector

The expected value $\mathrm{E}\, h(X, Y)$ of a real function h of random variables X and Y can be computed without determining the distribution of the variable $h(X, Y)$.

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  if the sum converges absolutely.

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- For X and Y continuous random variables it holds that

$$\mathrm{E}\, h(X,Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x,y) f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

  if the integral converges absolutely.

Properties of the expected value

Now we can prove the linearity of the expectation.

Theorem – linearity of expectation

For all $a, b \in \mathbb{R}$ and all random variables X and Y it holds that

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}\,X + b\,\mathrm{E}\,Y.$$

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Consequence:

- $\mathrm{E}(aX + b) = a\,\mathrm{E}\,X + b$. This statement was proven before separately.

Properties of the expected value

Proof

From the theory concerning the marginal distributions of discrete random variables X and Y we have:

$$\begin{aligned}
\mathrm{E}(aX+bY) &= \sum_{i,j}(ax_i+by_j)\,\mathrm{P}(X=x_i\cap Y=y_j)\\
&= \sum_{i,j} ax_i\,\mathrm{P}(X=x_i\cap Y=y_j)+\sum_{i,j} by_j\,\mathrm{P}(X=x_i\cap Y=y_j)\\
&= a\sum_i x_i\sum_j \mathrm{P}(X=x_i\cap Y=y_j)+b\sum_j y_j\sum_i \mathrm{P}(X=x_i\cap Y=y_j)\\
&= a\sum_i x_i\,\mathrm{P}(X=x_i)+b\sum_j y_j\,\mathrm{P}(Y=y_j) \quad = \quad a\,\mathrm{E}\,X+b\,\mathrm{E}\,Y.
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&= \sum_{i,j} ax_i\,\mathrm{P}(X=x_i\cap Y=y_j) + \sum_{i,j} by_j\,\mathrm{P}(X=x_i\cap Y=y_j) \\
&= a\sum_i x_i \sum_j \mathrm{P}(X=x_i\cap Y=y_j) + b\sum_j y_j \sum_i \mathrm{P}(X=x_i\cap Y=y_j) \\
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\end{aligned}$$

For continuous X and Y the proof is analogous:

$$\begin{aligned}
\mathrm{E}(aX+bY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax+by)f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \cdots = \\
&= a\int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y \quad = \quad a\,\mathrm{E}\,X + b\,\mathrm{E}\,Y.
\end{aligned}$$

□

Covariance and correlation coefficient

Mutual linear dependence of two random variables X and Y can be described in the following way:

Definition

Let X and Y be random variables with finite second moments. Then we define the **covariance** of the random variables X and Y as

$$\operatorname{cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}\, X)(Y - \mathrm{E}\, Y)].$$

If X and Y have positive variances then we define the **correlation coefficient** (or **coefficient of correlation**) as

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}}.$$

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Definition

Two random variables X and Y are called **non-correlated** if $\operatorname{cov}(X,Y) = 0$.

Covariance and the correlation coefficient – properties

Theorem

For the covariance and the correlation coefficient the following properties hold:

i) $\operatorname{cov}(X, Y) = \operatorname{E} XY - \operatorname{E} X \operatorname{E} Y,$

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iv) $\rho(aX + b, cY + d) = \rho(X, Y)$ *for all* $a, c > 0$ *and* $b, d \in \mathbb{R}$,

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For the covariance and the correlation coefficient the following properties hold:

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- **ii)** X *and* Y *are non-correlated if and only if* $\mathrm{E}\,XY = \mathrm{E}\,X\,\mathrm{E}\,Y$,
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Proof

i)
$$\begin{aligned}\operatorname{cov}(X,Y) &= \mathrm{E}\,((X-\mathrm{E}\,X)(Y-\mathrm{E}\,Y)) = \mathrm{E}\,(XY - X\,\mathrm{E}\,Y - Y\,\mathrm{E}\,X + \mathrm{E}\,X\,\mathrm{E}\,Y)\\ &= \mathrm{E}\,XY - \mathrm{E}(X\,\mathrm{E}\,Y) - \mathrm{E}(Y\,\mathrm{E}\,X) + \mathrm{E}(\mathrm{E}\,X\,\mathrm{E}\,Y)\\ &= \mathrm{E}\,XY - \mathrm{E}\,X\,\mathrm{E}\,Y - \mathrm{E}\,Y\,\mathrm{E}\,X + \mathrm{E}\,X\,\mathrm{E}\,Y = \mathrm{E}\,XY - \mathrm{E}\,X\,\mathrm{E}\,Y\end{aligned}$$

ii) Obvious from above.

iii) From the Schwarz inequality (see bibliography).

iv) Follows straightforwardly by inserting into the definition.

v) Follows from the proof of the Schwarz inequality (see bibliography). □

Non-correlated random variables

Let us study the expectation of the product XY of two random variables X and Y.

Definition

Alternative definition: Two random variables X and Y are called **non-correlated** if

$$\mathrm{E}\,XY = \mathrm{E}\,X\,\mathrm{E}\,Y.$$

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Lemma

If X and Y are independent then they are non-correlated.

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Lemma

If X and Y are independent then they are non-correlated.

Proof

Let X, Y be continuous variables. Independence means that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. Thus we have

$$\begin{aligned}\mathrm{E}\, XY &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y) \,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_X(x) f_Y(y) \,\mathrm{d}x\,\mathrm{d}y \\ &= \left(\int_{-\infty}^{+\infty} x f_X(x) \,\mathrm{d}x\right) \left(\int_{-\infty}^{+\infty} y f_Y(y) \,\mathrm{d}y\right) = \mathrm{E}\, X \,\mathrm{E}\, Y.\end{aligned}$$

□

Properties of the variance

It is now possible to obtain the following properties of the variance of sums of two random variables.

Theorem

i) *For X and Y with finite second moments:*

$$\operatorname{var}(X \pm Y) = \operatorname{var} X + \operatorname{var} Y \pm 2 \operatorname{cov}(X, Y).$$

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i) *For X and Y with finite second moments:*

$$\operatorname{var}(X \pm Y) = \operatorname{var} X + \operatorname{var} Y \pm 2 \operatorname{cov}(X, Y).$$

ii) *For non-correlated (independent) random variables it holds that*

$$\operatorname{var}(X \pm Y) = \operatorname{var} X + \operatorname{var} Y.$$

Properties of variance

Proof

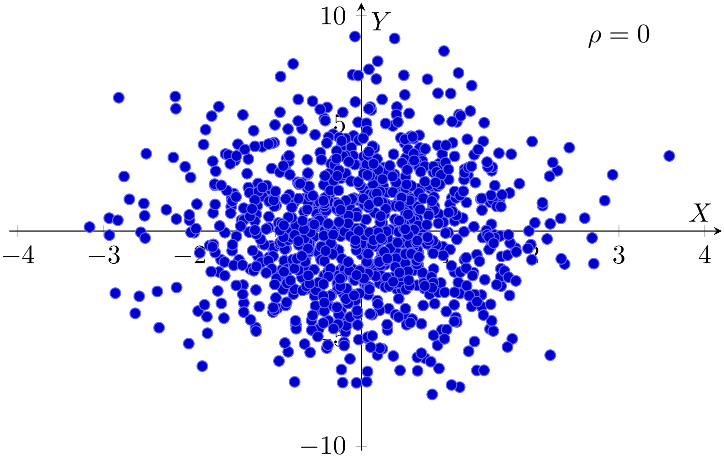
i) Given two random variables X and Y we have:

$$\begin{aligned}\operatorname{var}(X \pm Y) &= \operatorname{E}(X \pm Y)^2 - (\operatorname{E}(X \pm Y))^2 = \operatorname{E}(X^2 \pm 2XY + Y^2) - (\operatorname{E} X \pm \operatorname{E} Y)^2 \\ &= \operatorname{E} X^2 \pm 2 \operatorname{E} XY + \operatorname{E} Y^2 - (\operatorname{E} X)^2 \mp 2 \operatorname{E} X \operatorname{E} Y - (\operatorname{E} Y)^2 \\ &= \operatorname{var} X + \operatorname{var} Y \pm (2 \operatorname{E} XY - 2 \operatorname{E} X \operatorname{E} Y) = \operatorname{var} X + \operatorname{var} Y \pm 2 \operatorname{cov}(X, Y).\end{aligned}$$

ii) For non-correlated (independent) random variables the covariance is zero.

□

Correlation – sample of 1000 values



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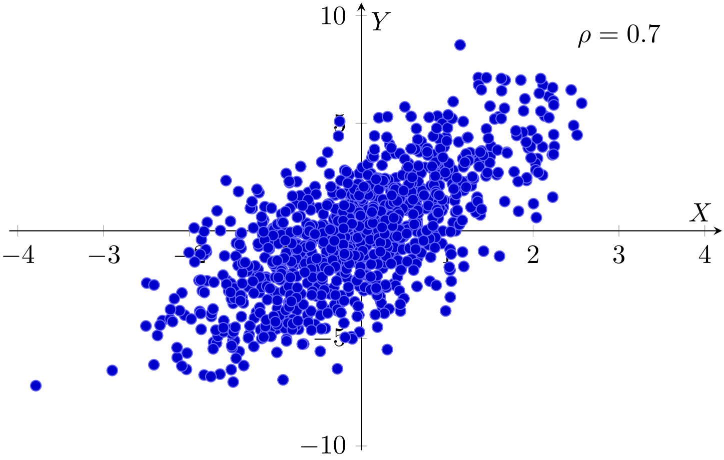

Sums of random variables

An important case of a function of multiple random variables is their sum

$$Z = h(\boldsymbol{X}) = h(X_1, \ldots, X_n) = X_1 + \cdots + X_n.$$

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Consider for simplicity a sum of two random variables:

- If X and Y are **discrete** and **independent**, then for $Z = X + Y$ it holds that

$$\mathrm{P}(Z = z) = \sum_x \mathrm{P}(X = x) \cdot \mathrm{P}(Y = z - x) \quad \text{(\textbf{discrete convolution})}.$$

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$$\mathrm{P}(Z = z) = \sum_x \mathrm{P}(X = x) \cdot \mathrm{P}(Y = z - x) \quad \text{(\textbf{discrete convolution})}.$$

- If X and Y are **continuous** and **independent**, then for $Z = X + Y$ it holds that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) \, \mathrm{d}x \quad \text{(\textbf{convolution of } } \boldsymbol{f_X} \text{ \textbf{and} } \boldsymbol{f_Y}).$$

Sums of random variables – convolution (discrete case)

The expression for the sum of discrete independent X and Y is obtained easily:

$$\begin{aligned} \mathrm{P}(Z = z) &= \mathrm{P}(X + Y = z) \\ &= \sum_{\{(x_k, y_j):\, x_k + y_j = z\}} \mathrm{P}(X = x_k \cap Y = y_j) \\ &= \sum_{\text{all } x_k} \mathrm{P}(X = x_k)\ \mathrm{P}(Y = z - x_k). \end{aligned}$$

Sums of random variables - convolution (continuous case)

For continuous independent X and Y we have:

$$
\begin{aligned}
F_Z(z) = \mathrm{P}(X+Y \le z) &= \iint\limits_{\{(x,y):\, x+y\le z\}} f_{X,Y}(x,y)\,\mathrm{d}(x,y) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,\mathrm{d}y \right) \mathrm{d}x \\
&\overset{y=u-x}{=} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z} f_{X,Y}(x,u-x)\,\mathrm{d}u \right) \mathrm{d}x \\
&= \int_{-\infty}^{z} \left(\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,\mathrm{d}x \right) \mathrm{d}u \\
&= \int_{-\infty}^{z} \left(\int_{-\infty}^{\infty} f_X(x) f_Y(u-x)\,\mathrm{d}x \right) \mathrm{d}u.
\end{aligned}
$$

The density f_Z is any non-negative function, for which $F_Z(z) = \int_{-\infty}^{z} f_Z(u)\,\mathrm{d}u$.
The expression under the first integral $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,\mathrm{d}x$ is thus the density of Z.

Sum of random variables – Normal distribution

Example – sum of two normal distributions

Suppose that X and Y are independent, both having the normal distribution N$(\mu, 1)$. We want to obtain the distribution of $Z = X + Y$.

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The densities of X and Y correspond to the normal distribution with variance $\sigma^2 = 1$:

$$f_X(x) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{(x-\mu)^2}{2 \cdot 1}}, \qquad f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{(y-\mu)^2}{2 \cdot 1}} \qquad x, y \in \mathbb{R}.$$

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Suppose that X and Y are independent, both having the normal distribution $\mathrm{N}(\mu, 1)$. We want to obtain the distribution of $Z = X + Y$.

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The density of the sum is obtained using convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x-\mu)^2}{2}}\,\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}\left((x-\mu)^2 + (z-x-\mu)^2\right)}\,\mathrm{d}x.$$

Sum of random variables – Normal distribution

Example – sum of two normal distributions, continuation

The expressions in the exponent can be rewritten as:

$$(x-\mu)^2 + (z-x-\mu)^2 = x^2 - 2\mu x + \mu^2 + z^2 + x^2 + \mu^2 - 2zx - 2\mu z + 2\mu x$$
$$= 2\left(x - \frac{z}{2}\right)^2 + \frac{1}{2}(z-2\mu)^2 .$$

The expression under the integral can then be split into two multiplicative parts, with one of them not depending on x and the other one having an integral of 1:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{2(x-z/2)^2}{2}} e^{-\frac{(z-2\mu)^2}{2\cdot 2}}\,\mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi 2}} e^{-\frac{(z-2\mu)^2}{2\cdot 2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1/2)}} e^{-\frac{(x-z/2)^2}{2\cdot(1/2)}}\,\mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi 2}} e^{-\frac{(z-2\mu)^2}{2\cdot 2}} .$$

The sum $Z = X + Y$ has therefore the normal distribution $\mathrm{N}(2\mu, 2)$. In general, it can be proven that the sum of n independent normals $\mathrm{N}(\mu, \sigma^2)$ has the distribution $\mathrm{N}(n\mu, n\sigma^2)$.

Sum of random variables – Poisson distribution

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From what we have seen before we know that for $k = 0, 1, \ldots$:

$$\mathrm{P}(Z = k) = \sum_{\{(j,\ell)\in\mathbb{N}_0^2 : \, j+\ell=k\}} \mathrm{P}(X = j)\,\mathrm{P}(Y = \ell) = \sum_{i=0}^{k} \mathrm{P}(X = j)\,\mathrm{P}(Y = k - j)$$

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$$= \sum_{j=0}^{k} \frac{\lambda_1^j}{j!} e^{-\lambda_1} \frac{\lambda_2^{k-j}}{(k-j)!} e^{-\lambda_2} = e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda_1^j \lambda_2^{k-j}$$

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$$= \sum_{j=0}^{k} \frac{\lambda_1^j}{j!} e^{-\lambda_1} \frac{\lambda_2^{k-j}}{(k-j)!} e^{-\lambda_2} = e^{-(\lambda_1 + \lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda_1^j \lambda_2^{k-j}$$

$$= \frac{(\lambda_1 + \lambda_2)^k}{k!} e^{-(\lambda_1 + \lambda_2)}. \qquad \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

Sum of random variables – Poisson distribution

Example

Consider two independent random variables X and Y with the Poisson distribution with parameters λ_1 and λ_2, respectively. Find the distribution of the variable $Z = X + Y$.

$$\mathrm{P}(X = j) = \frac{\lambda_1^j}{j} e^{-\lambda_1} \qquad \mathrm{P}(Y = \ell) = \frac{\lambda_2^\ell}{\ell} e^{-\lambda_2}, \qquad j, \ell = 0, 1, \dots$$

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✓ An easier way is to use the moment generating function.

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Then $M_{X_i}(s) = (1-p)e^{0s} + pe^{1s} = 1 - p + pe^s, \quad i = 1, \ldots, n.$

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Z is again a Poisson random variable, this time with the parameter $\lambda_1 + \lambda_2$:

$$\mathrm{P}(Z = k) = \frac{(\lambda_1 + \lambda_2)^k}{k!}e^{-(\lambda_1+\lambda_2)}.$$

Compare with the difficulty of a direct computation of the convolution.

Summary

Joint distribution function of a random vector (X, Y):

$$F_{X,Y}(x, y) = \mathrm{P}(X \le x \cap Y \le y).$$

Discrete random variables X and Y	**Continuous random variables X and Y**
Joint probabilities of values / density:	
$\mathrm{P}(X = x \cap Y = y)$	$f_{X,Y}(x, y)$
Marginal probabilities / density of X:	
$\mathrm{P}(X = x) = \sum_y \mathrm{P}(X = x \cap Y = y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$
$\mathrm{P}(Y = y) = \sum_x \mathrm{P}(X = x \cap Y = y)$	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x$
Independence of X and Y:	
$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y)$	$f_{X,Y}(x, y) = f_X(x) f_Y(y)$

Covariance of X and Y:

$$\mathrm{cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}\,X)(Y - \mathrm{E}\,Y)] = \mathrm{E}[XY] - \mathrm{E}\,X\,\mathrm{E}\,Y$$

X and Y are called non-correlated whenever $\mathrm{cov}(X, Y) = 0$.

If X and Y are independent, then they are also non-correlated.