Parameter point estimators

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Probability and Statistics

BIE-PST, WS 2024/25, Lecture 9

Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

- A **random variable** X is a measurable function, which assigns real values to the outcomes of a random experiment.
- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

$$F_X(x) = \mathrm{P}(X \leq x).$$

- Often we observe a **sequence of independent and identically distributed** (`i.i.d.`) random variables $X_1, X_2, \ldots$. Let each of them have expectation μ and variance σ^2.
- If we denote the **sum** and the arithmetic **mean** of n such variables as

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

we get that

$$\begin{aligned} \mathrm{E}\, S_n &= n \cdot \mu, & \mathrm{E}\, \bar{X}_n &= \mu, \\ \mathrm{var}\, S_n &= n \cdot \sigma^2, & \mathrm{var}\, \bar{X}_n &= \sigma^2 / n. \end{aligned}$$

- According to the **law of large numbers**, the arithmetic mean converges to the expectation, provided that it exists:

$$\bar{X}_n \overset{n \to \infty}{\longrightarrow} \mu.$$

Introduction to statistics

So far we have dealt with **probabilistic** problems with known parameters. For example if we have a box with r red and b blue balls, we can:

- find the probability of drawing a blue ball,
- find the probability of drawing a certain number of blue balls in three draws with or without replacement,
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Now we will deal with **statistical** problems. For example if we have a box with an unknown number of red and blue balls, we can take a sample and:

- estimate the proportion of red and blue balls,
- test whether there are 50% of blue balls or more,
- test whether the red/blue proportion is the same among two separate boxes,
- etc.

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Mathematical statistics proceeds, to some extent, reversely. On the grounds of real outcomes we choose an appropriate model and estimate its parameters. Then we can test hypotheses about these parameters and verify how well does the model fit the data.

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Statistics uses specific terminology.

Definition

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The **random sample realization** (random vector of observations or simply **data**) is an n-tuple of particular observed values $x_1, \ldots, x_n$.

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  - **Parametric tests** – we state a hypothesis about the parameter θ (e.g., $\theta = 0$) and on the grounds of measured data we try to decide whether this hypothesis can be true or not.

Estimation of the shape of the distribution – model selection

The distribution of an investigated random variable usually may not be absolutely arbitrary.

Based on previous experience, intuition or the type of underlying data we can often

- determine whether the variable is discrete or continuous;
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This information leads us to a choice of a particular **model**, thus to the

- choice of parametric distribution family $\{F_\theta(x)|\theta \in \Theta\}$, where Θ is a set of all possible values of parameter θ;
- and the assumption that our random sample is governed by distribution from this family.

Examples of possible models

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- **Normal distribution** – results of an IQ test in a given population

$$\{\mathrm{N}(\mu,\sigma^2) \mid \mu \in (-\infty,+\infty), \sigma^2 \in (0,+\infty)\}$$

  Two dimensional parameter $\theta = (\mu,\sigma^2)$ and $\Theta = (-\infty,+\infty) \times (0,+\infty)$.

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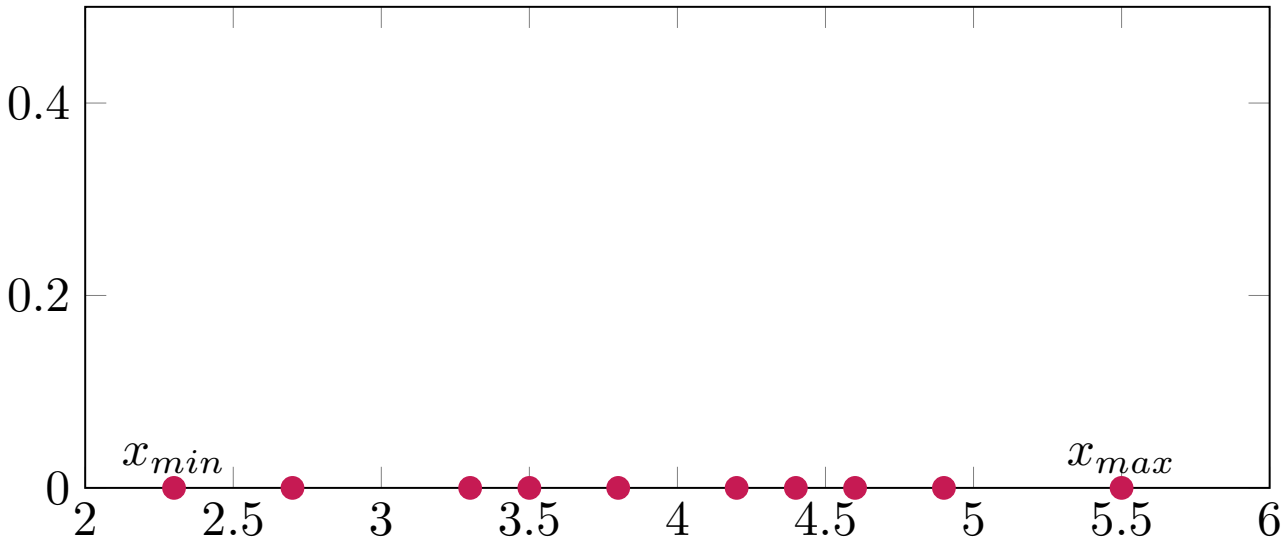

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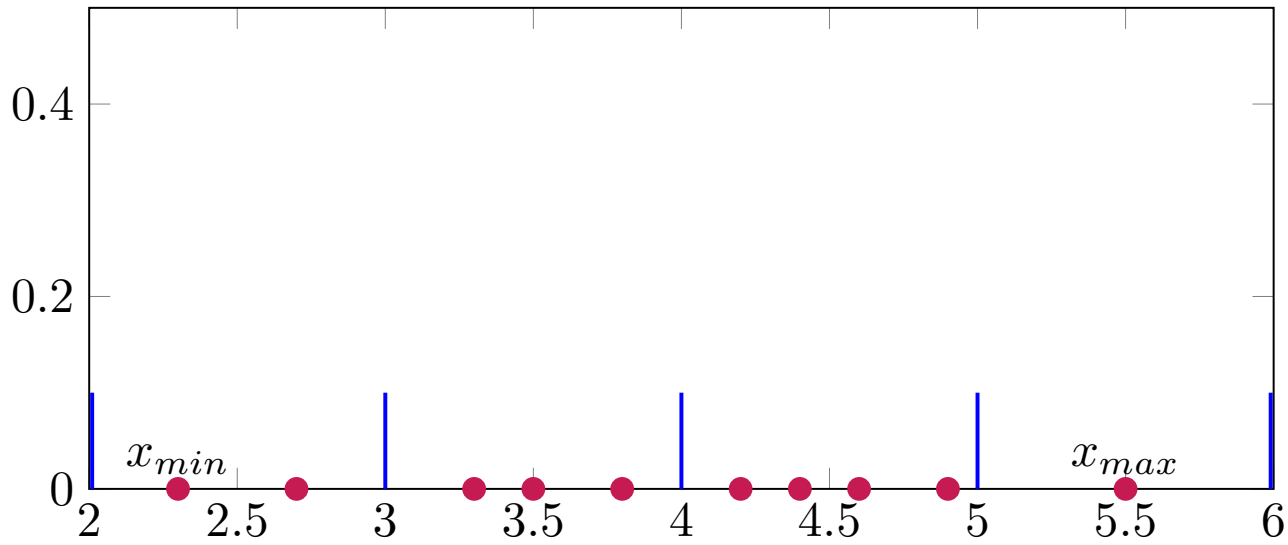

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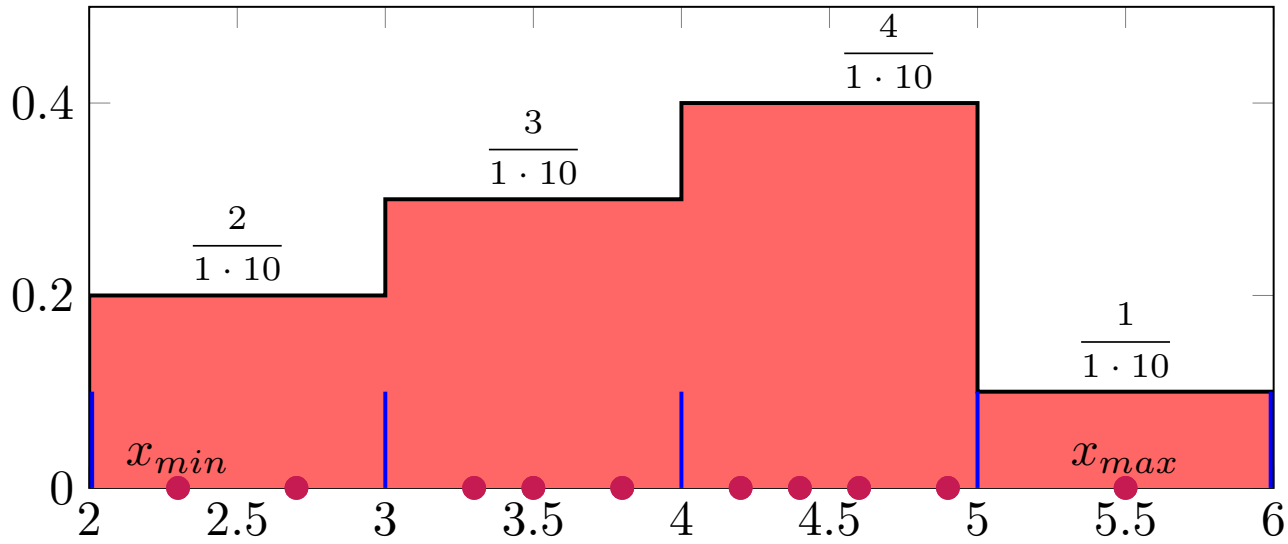

- Determine the data range.
- Choose a number of bins k and their size h (here $k = 4$ and $h = 1$).
- Over each bin, plot a column of the size

$$\frac{\text{number of observation in bin}}{h \cdot \text{total number of observations}} \overset{\text{denote}}{=} \frac{m_i}{h \cdot n}.$$

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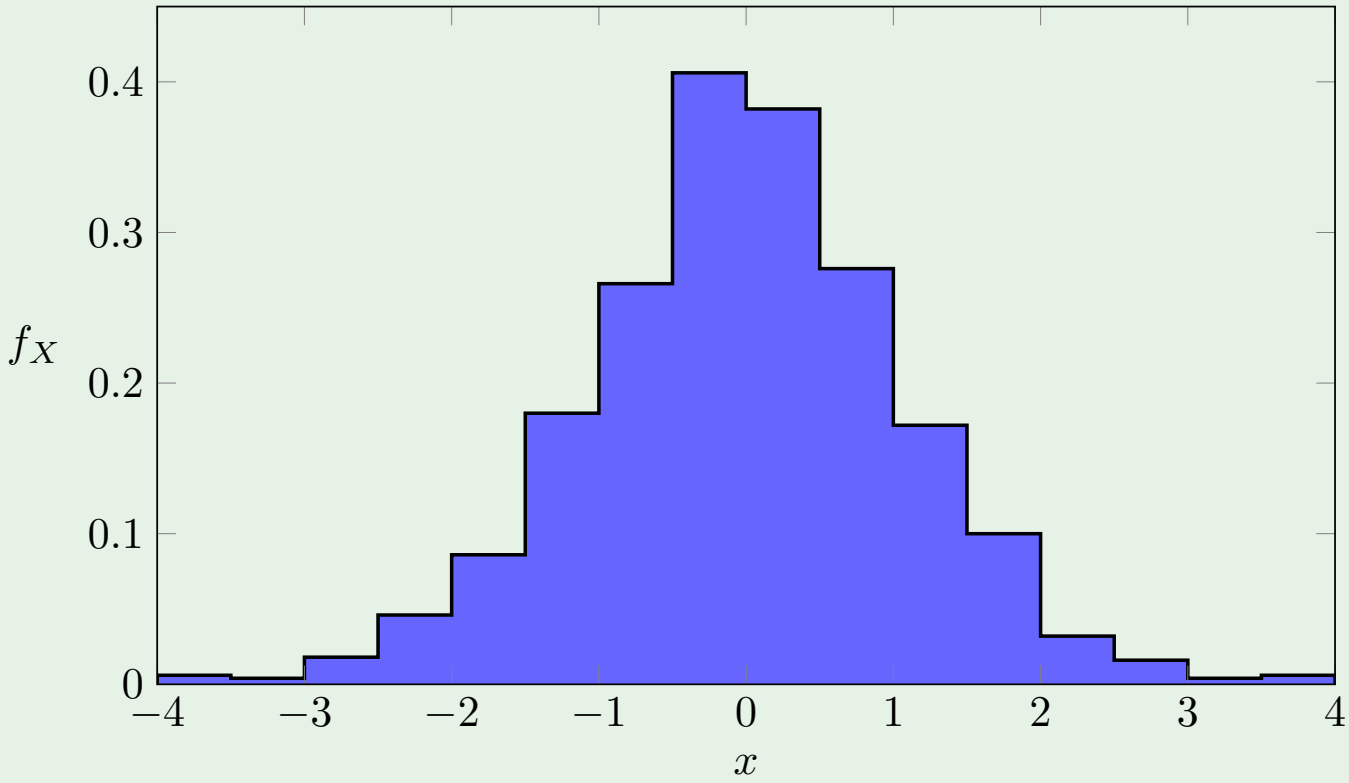
Example

We measured 1000 values from an unknown distribution. The histogram of these values is:



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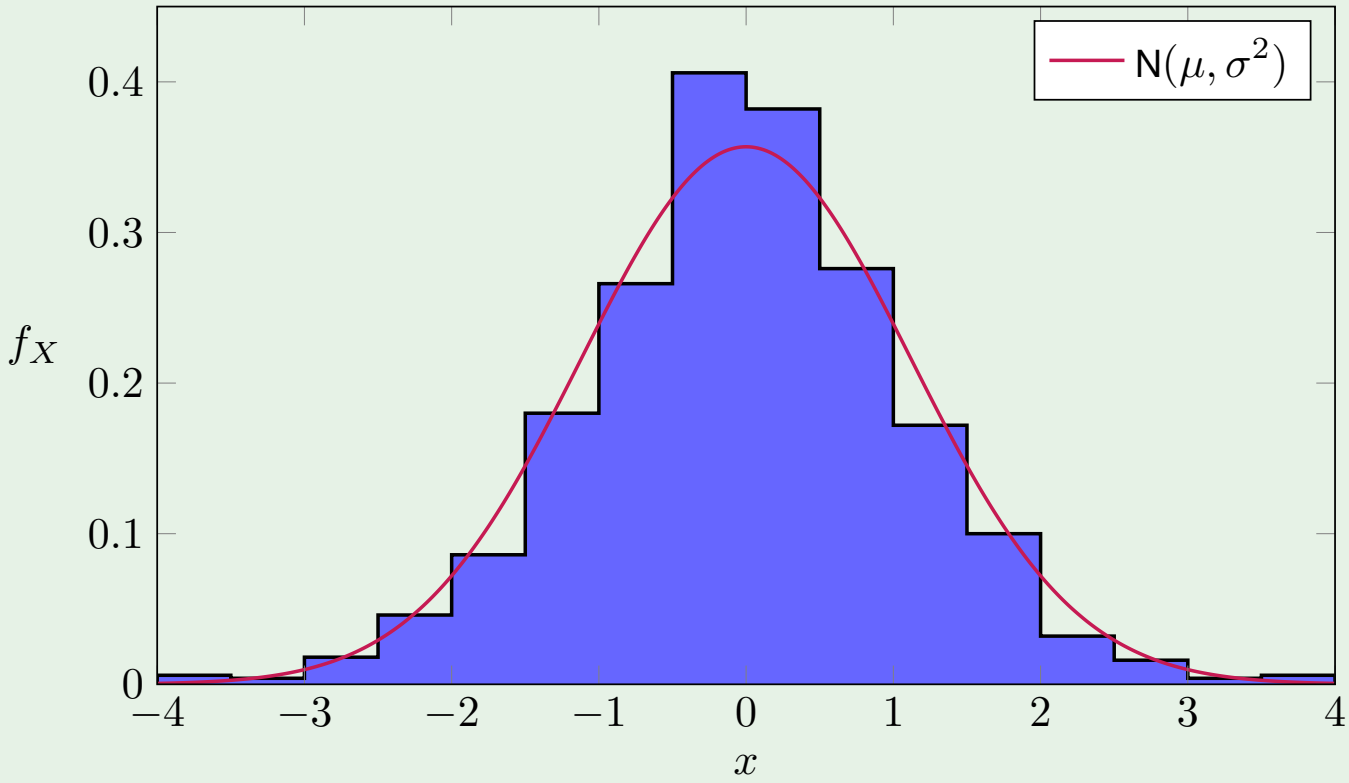

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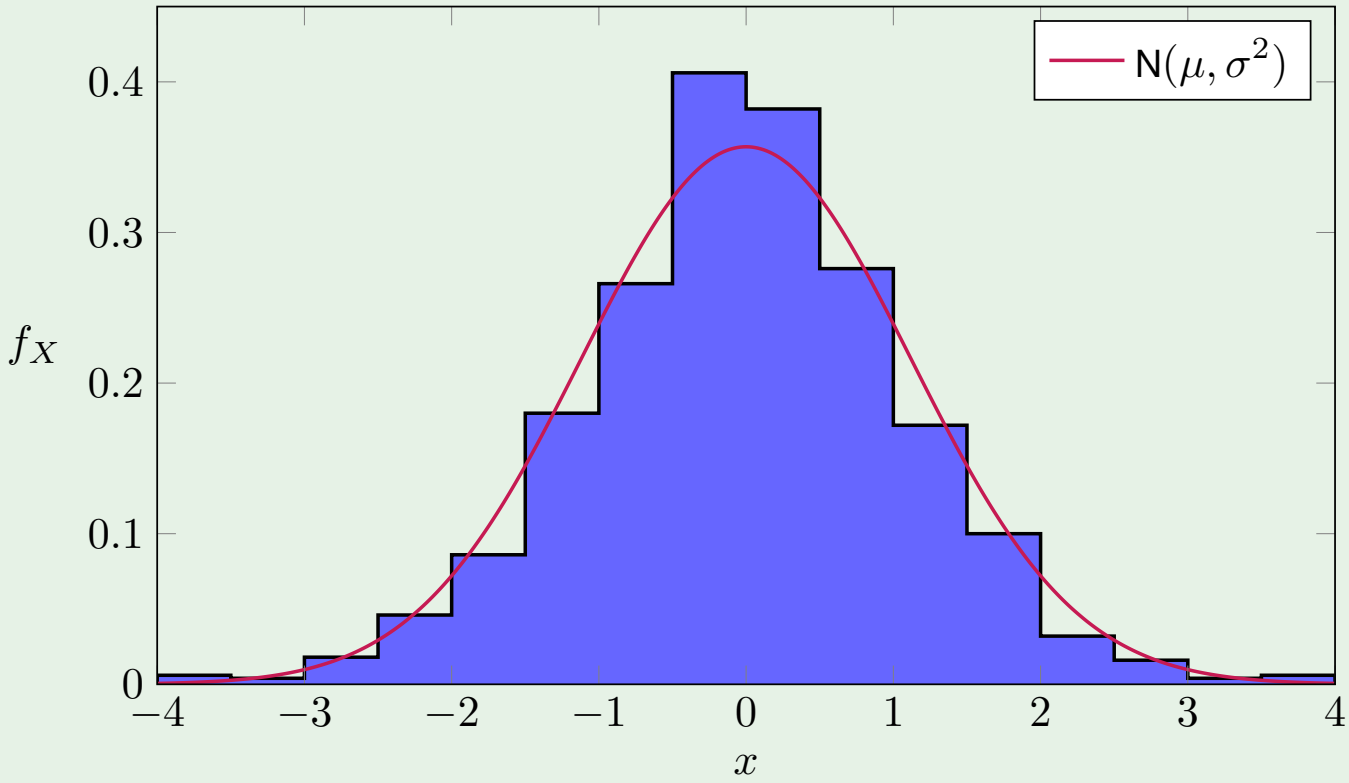

We can assume that we deal with values from the normal distribution with unknown parameters μ and σ^2.

Empirical distribution function

The shape of the **distribution function** can be estimated by the **empirical distribution function**:

$$F_n(x, X_1, \ldots, X_n) = F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}.$$

In other words, the probability that the random variable in question is less than or equal x can be estimated by the proportion of data points which are less than or equal to x.

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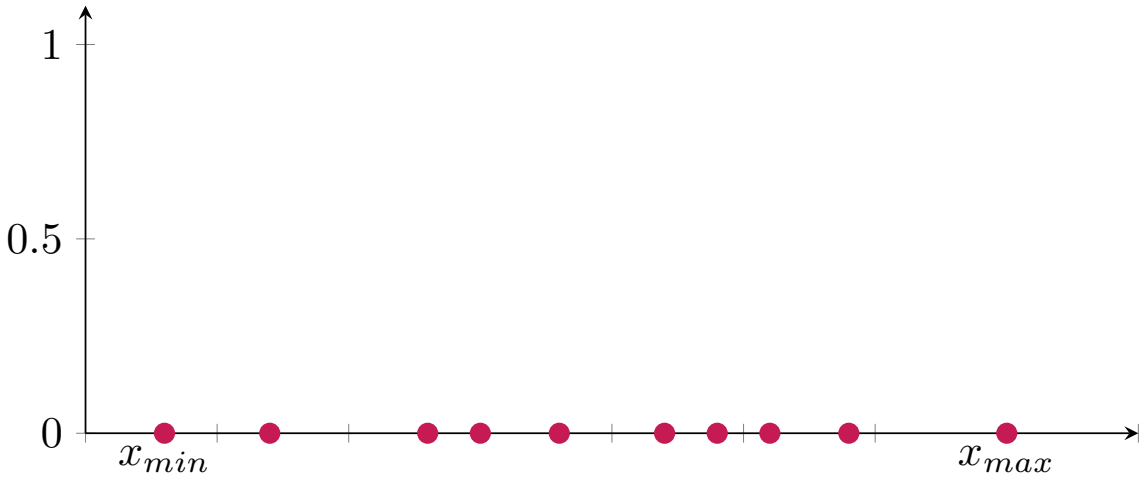

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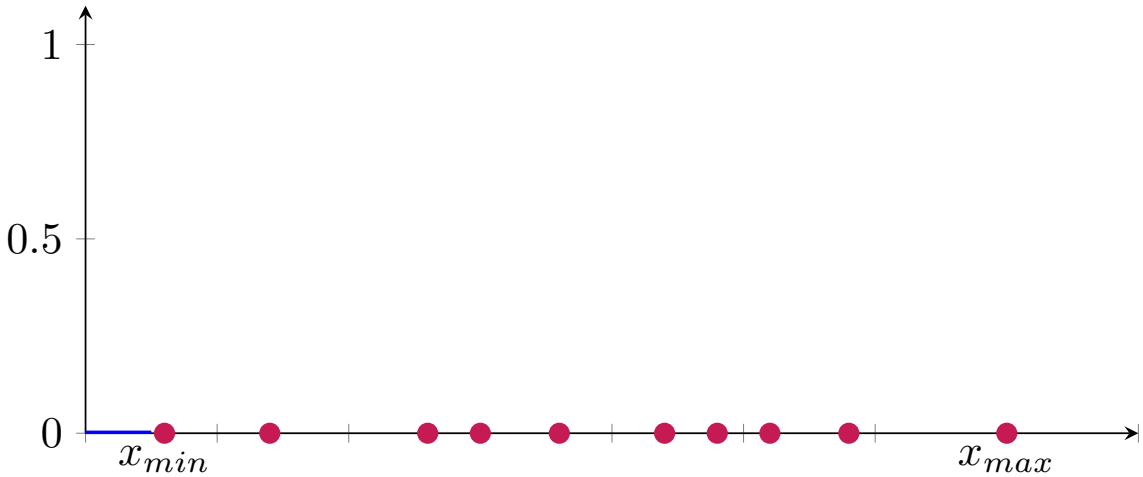

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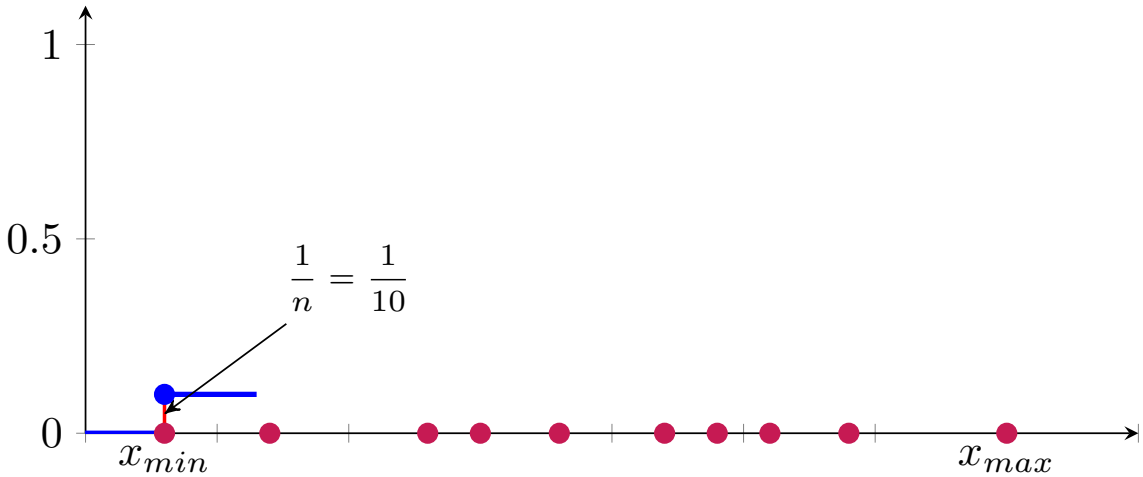

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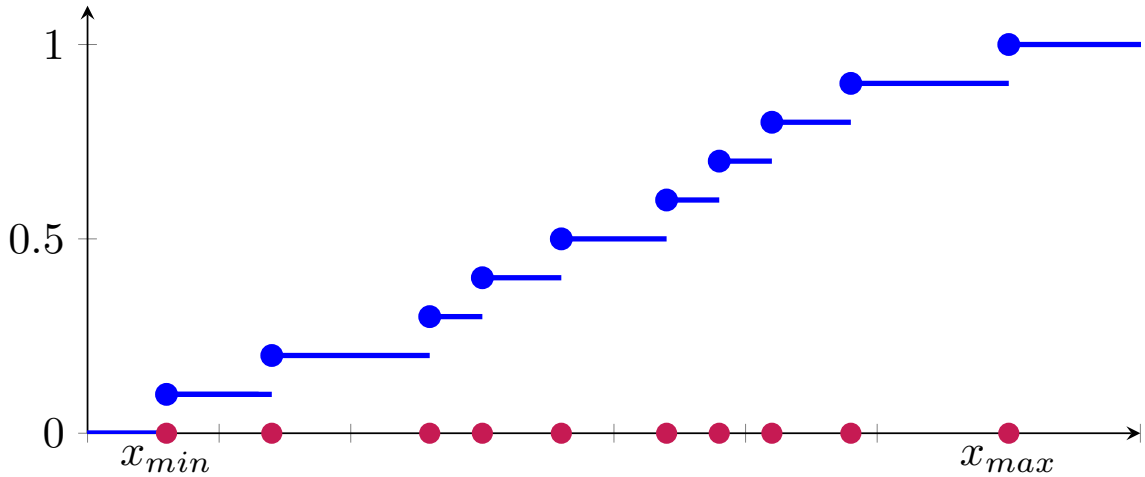

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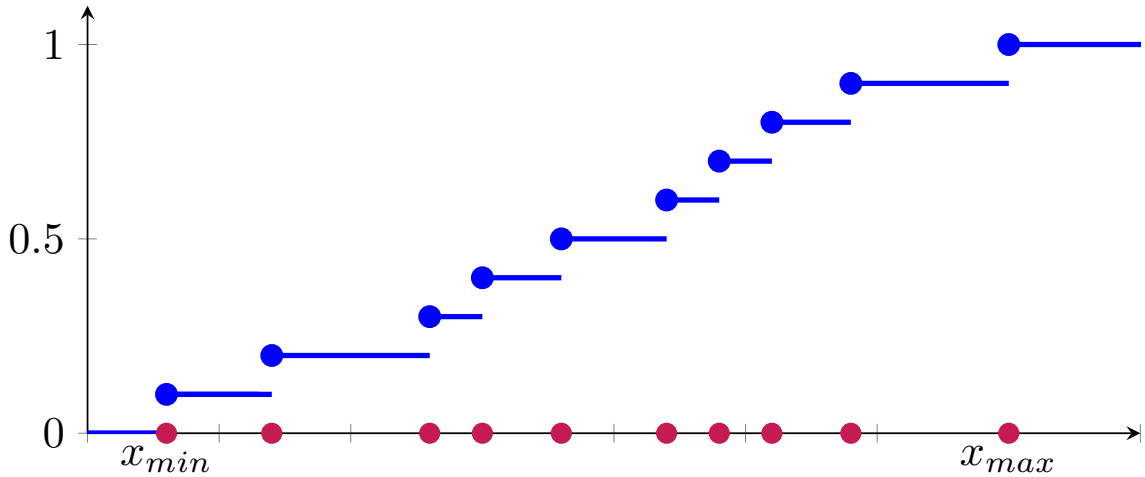

✓ The empirical distribution function is a piecewise constant function with jumps of size $\frac{1}{n}$ in the observed data points.

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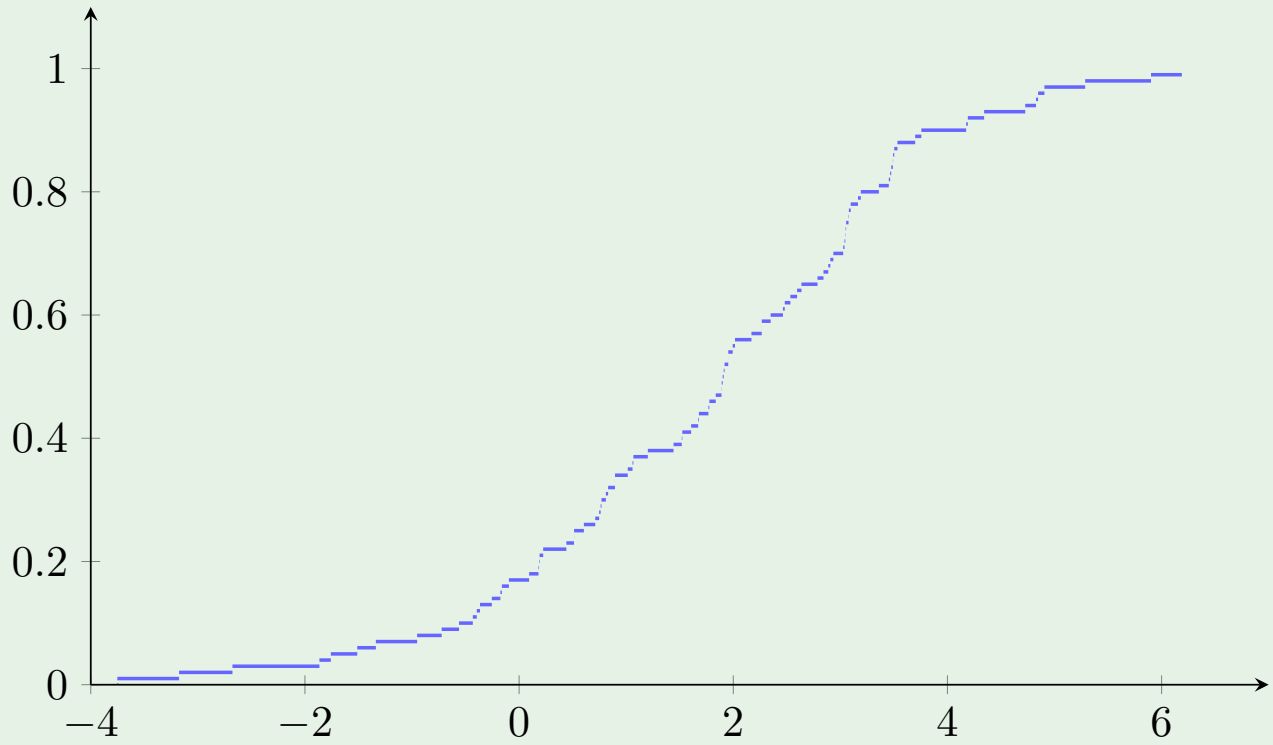

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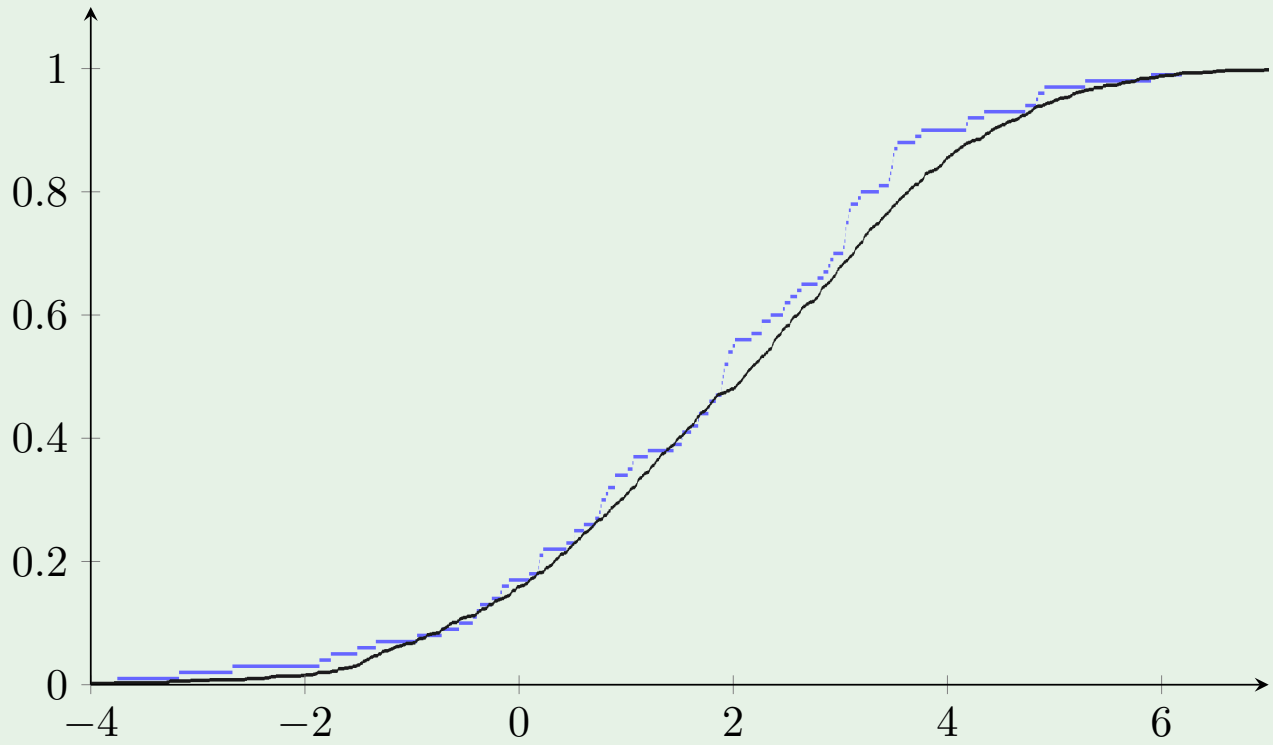

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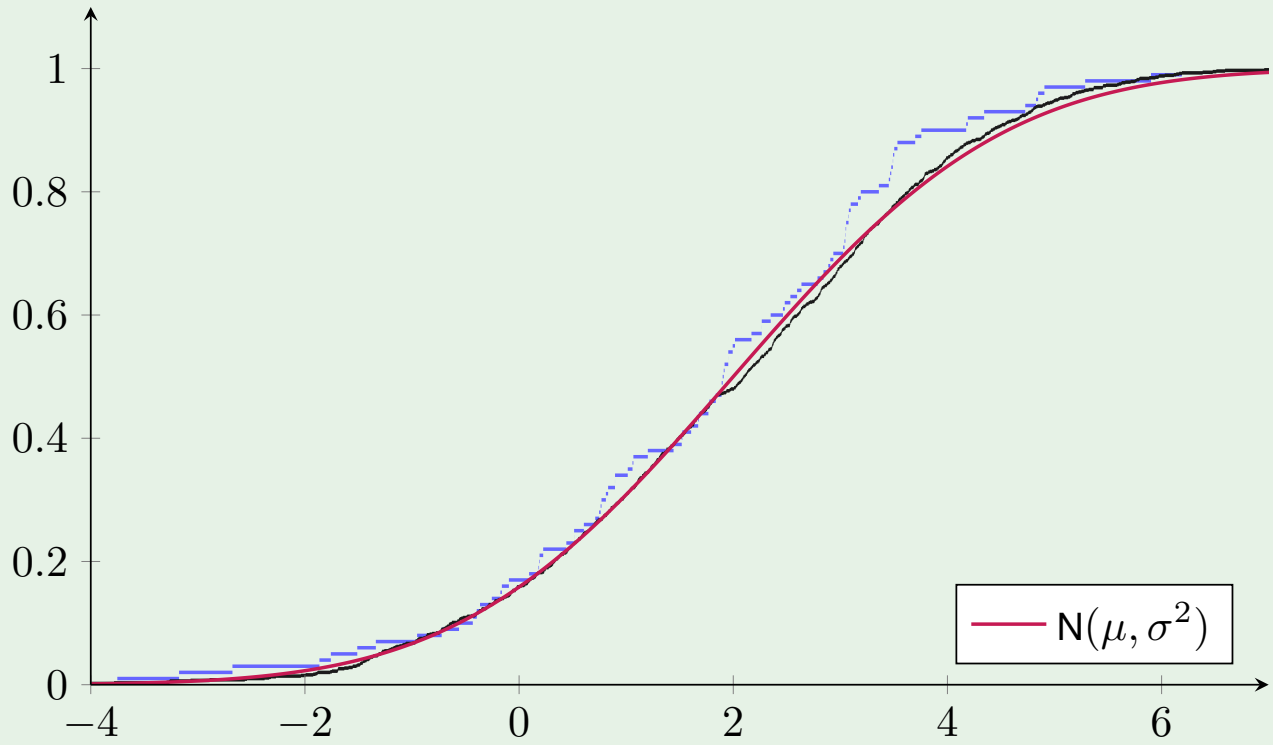

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Estimating the shape of the distribution – example

Example – waiting for a bus

Every morning we measure the time which we spend waiting for a bus on our way to school. After 15 days, we have observed the following data (in minutes, sorted):

0.1	0.3	0.5	0.7	1.0
1.9	2.8	3.4	3.5	3.8
5.3	7.7	8.6	8.7	11.1

Suppose that the waiting times form a random sample $(X_1, \ldots, X_{15})$ from an unknown distribution. *Find the histogram and the empirical distribution function of this distribution.*

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The data are in the interval $[0, 12]$. If we take the bandwidth h too small or too large, the histogram may be inaccurate:

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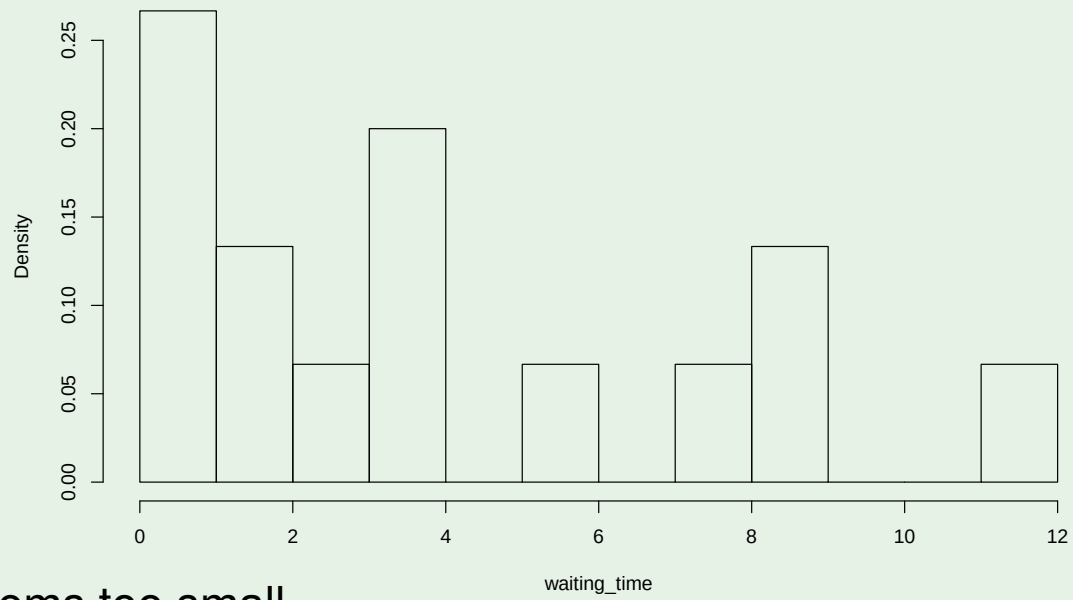

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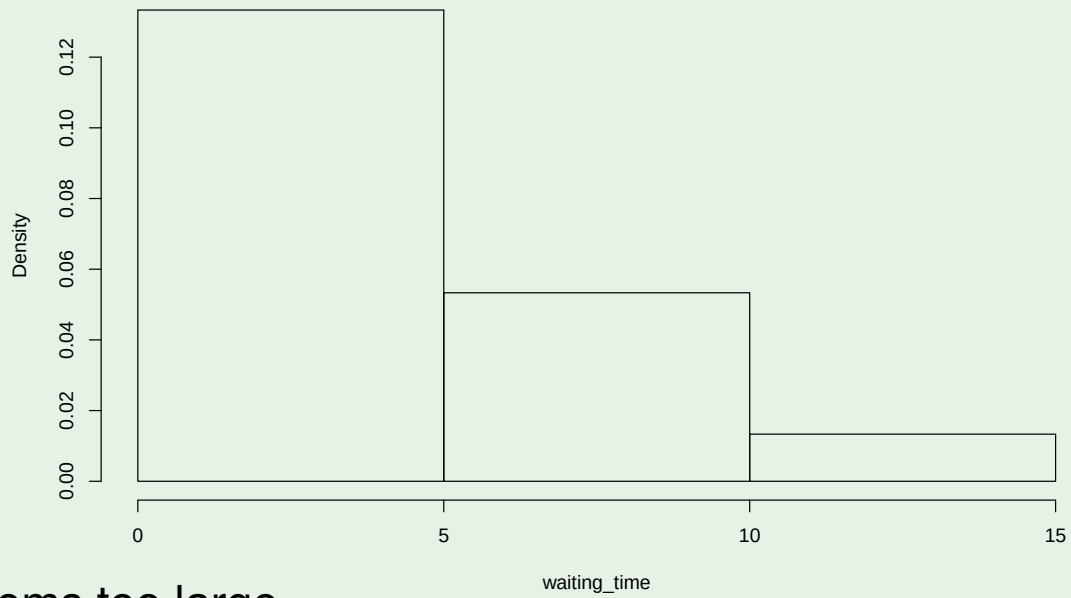

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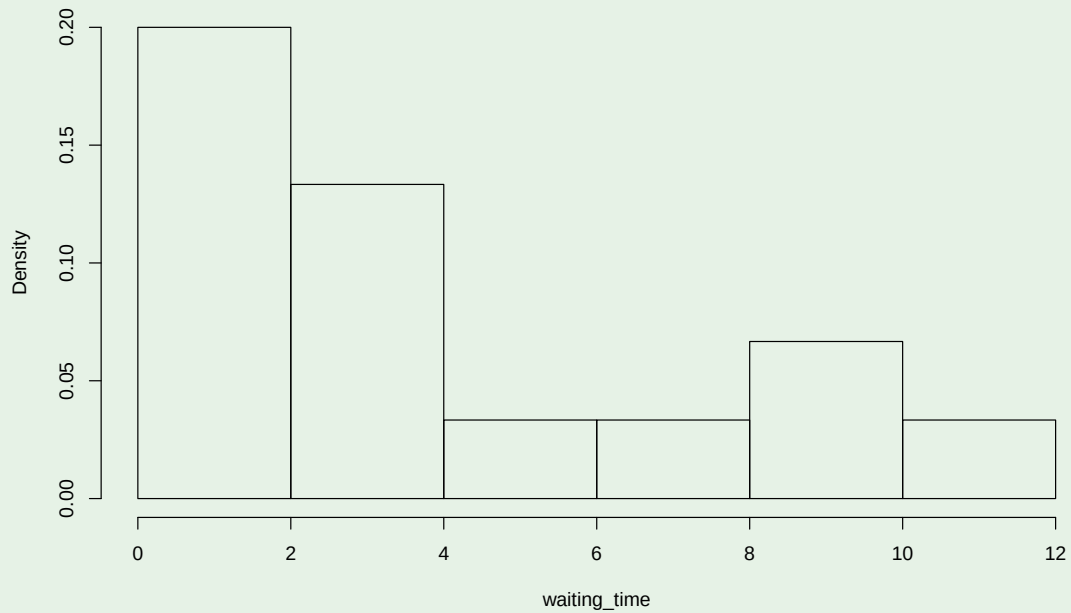

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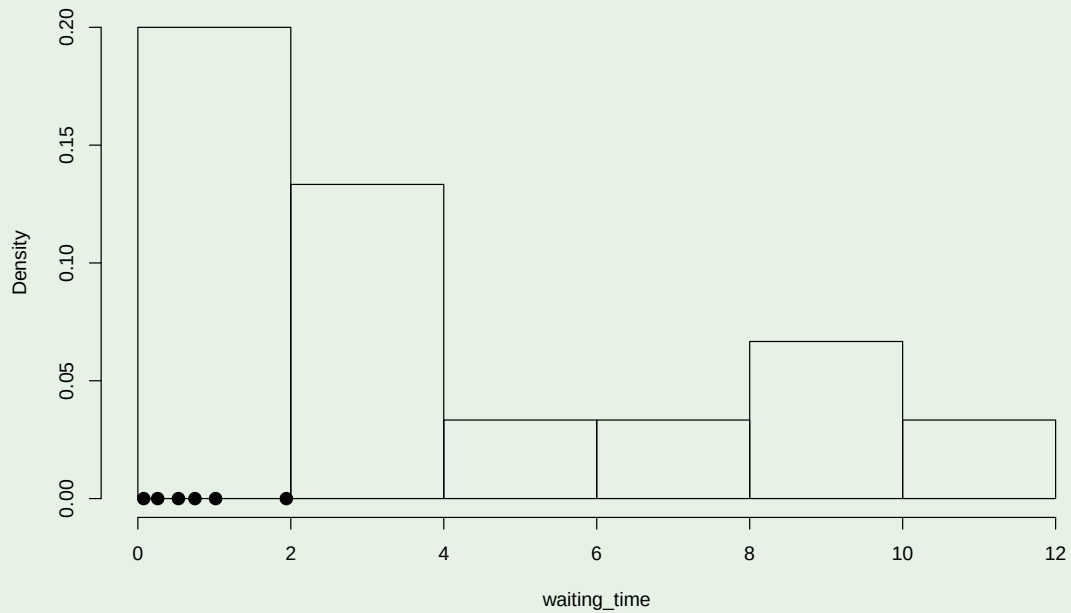

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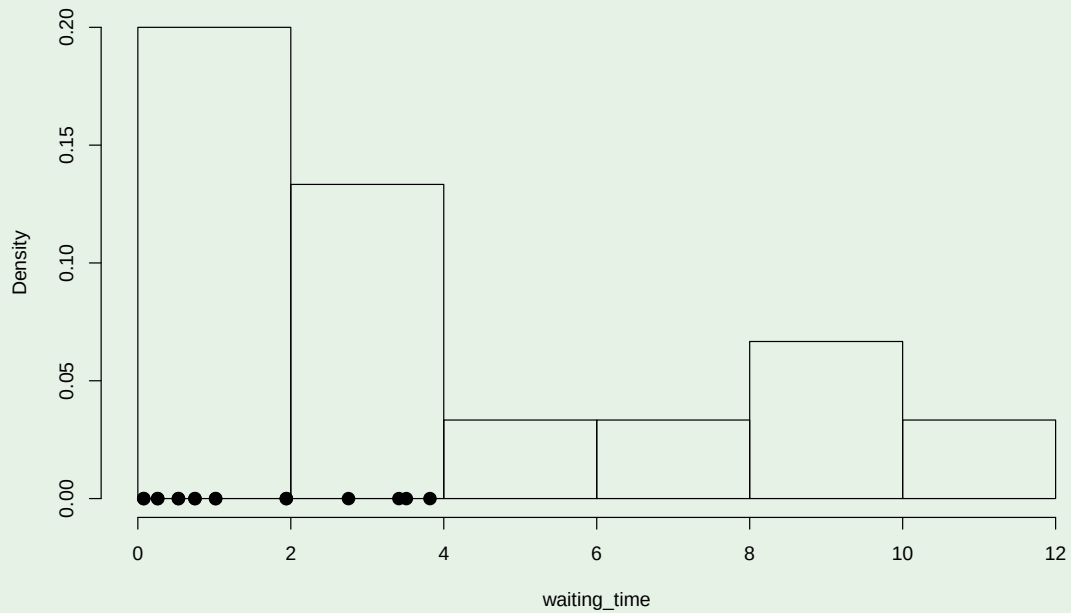

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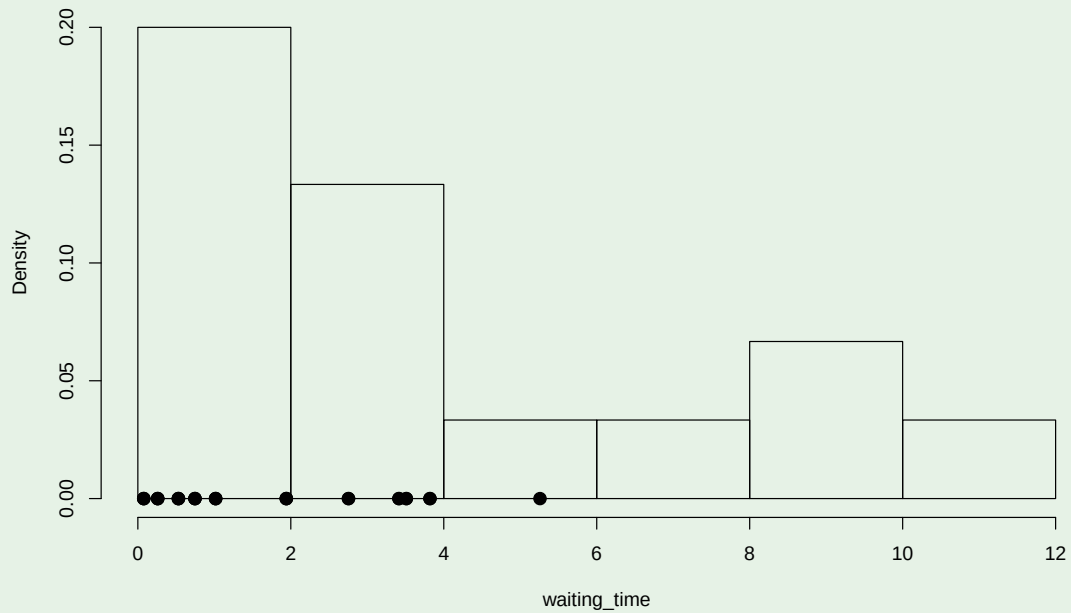

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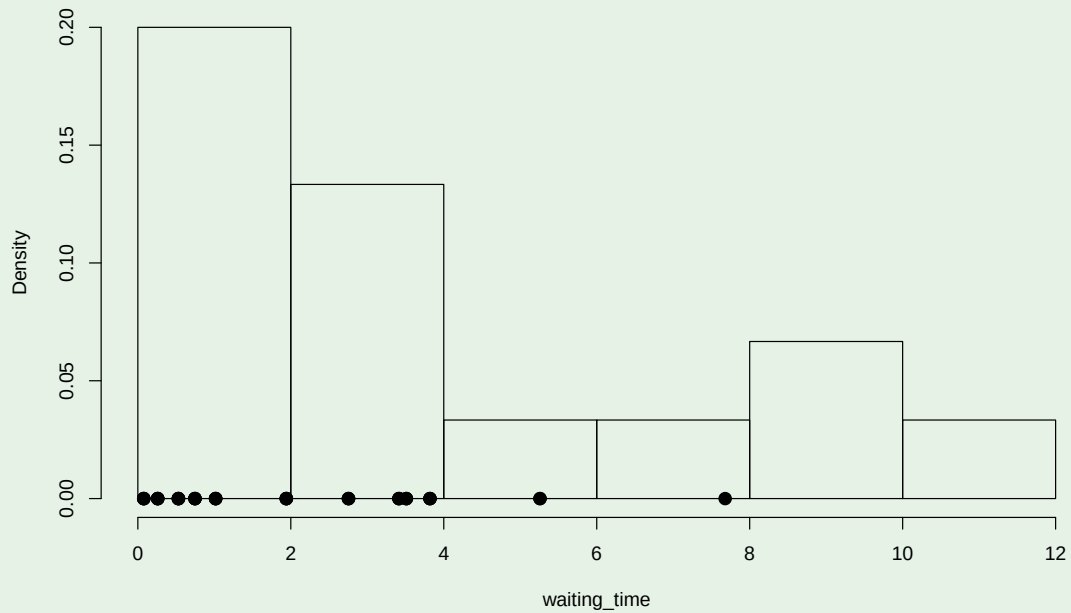

The histogram might seem similar to the exponential distribution.

Estimating the shape of the distribution – example

Example – waiting for a bus – histogram

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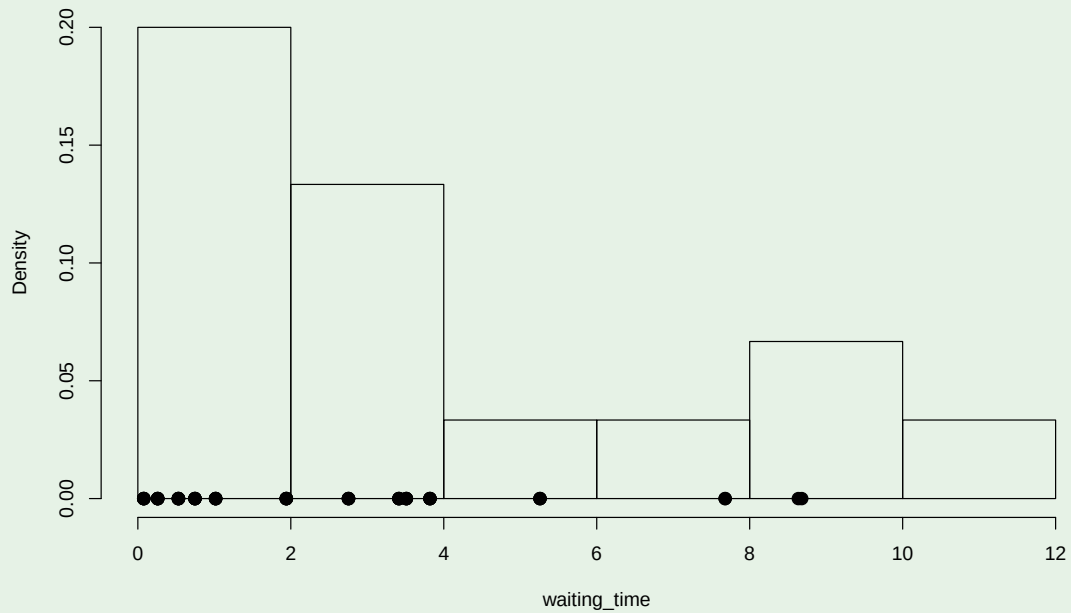

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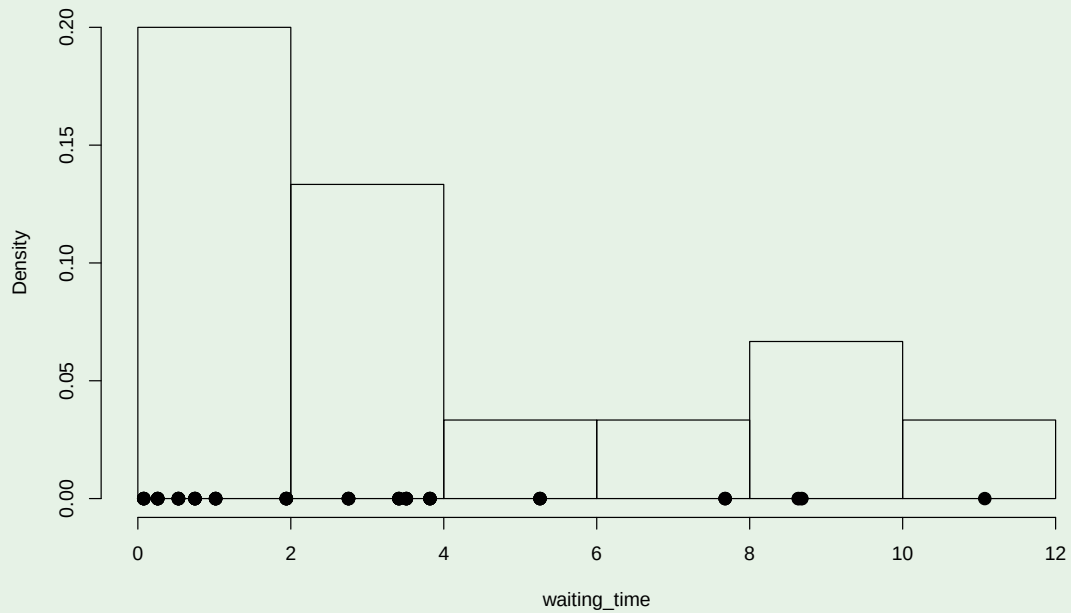

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We proceed from the left and add a jump of $1/15$ at each data point encountered:

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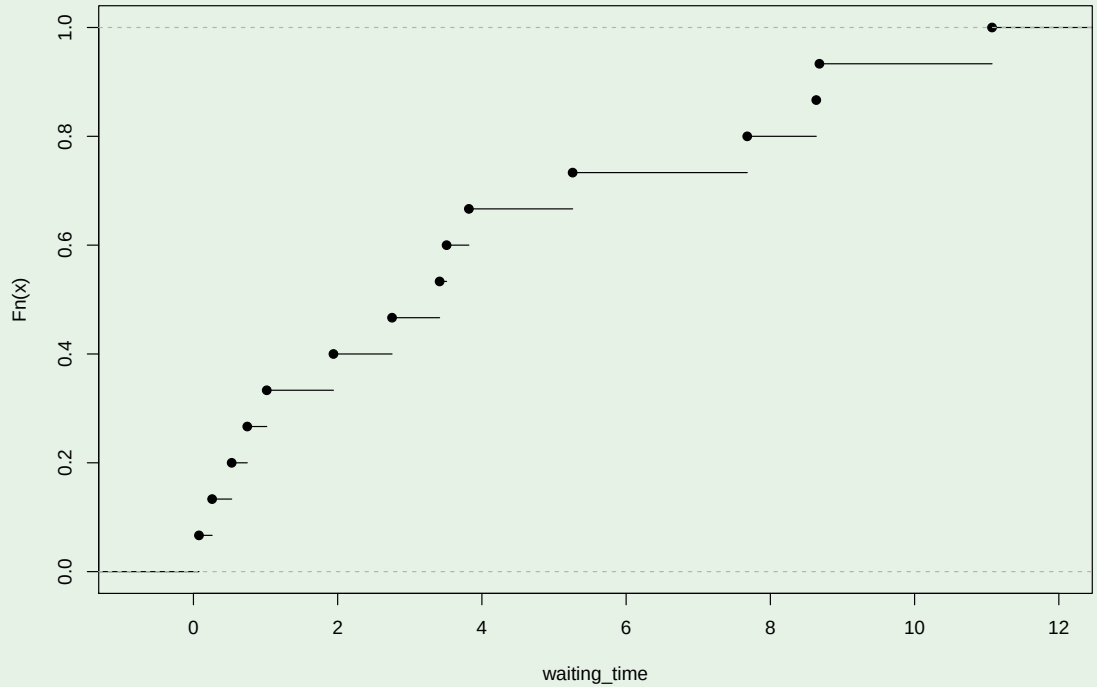

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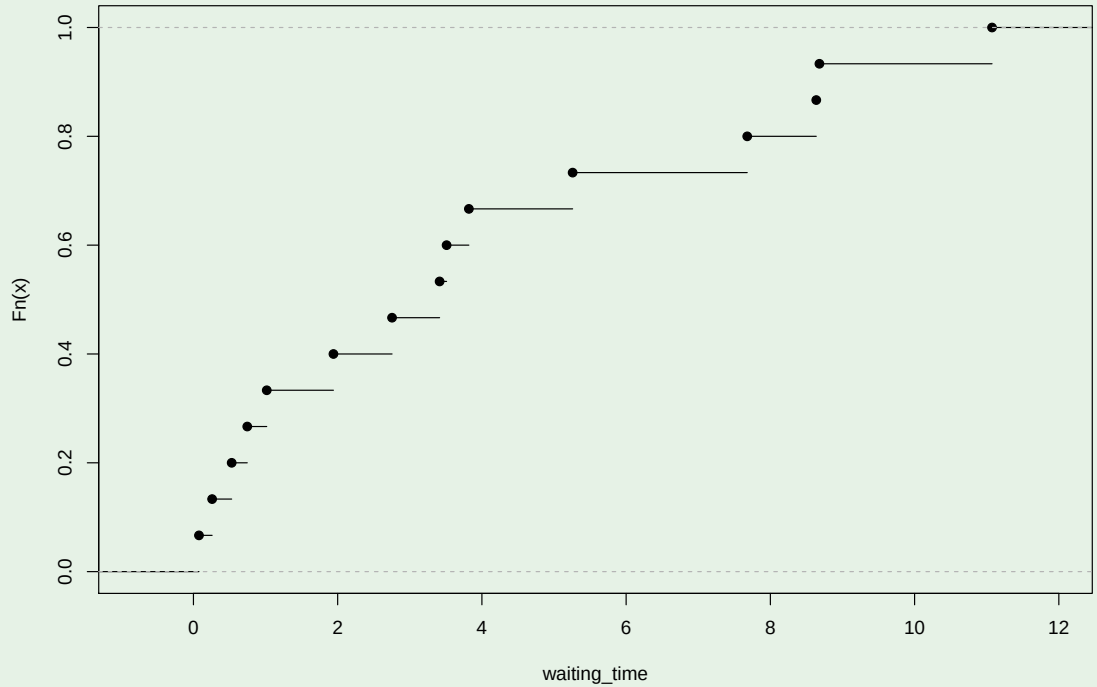

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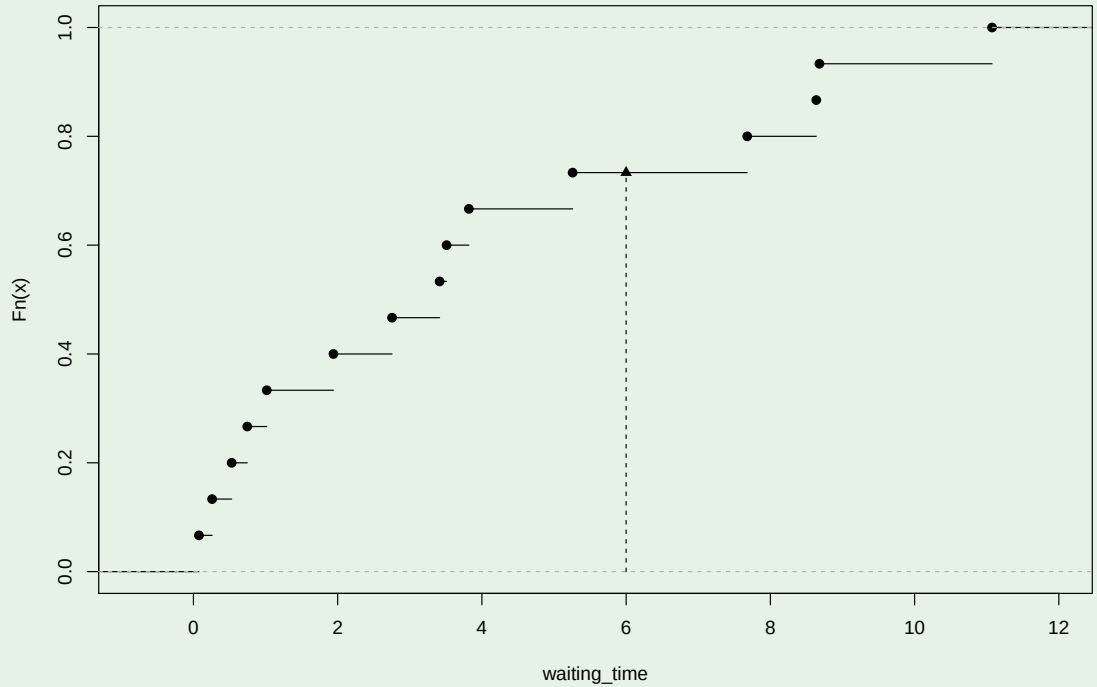

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The probability that we do not need to wait for more than six minutes is estimated as $F_n(6) = 11/15 \doteq 0.733$, which is the proportion of data points less than or equal to 6.

Estimating the shape of the distribution – quantiles

The **quantiles** q_α divide the population so that there are $\alpha\%$ of values under the α-quantile and $(1-\alpha)\%$ above. The 50%-quantile is called the **median** and divides the population into two equally large parts with respect to probability.

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Estimate the median of the time spent waiting for the bus using the observed data:

0.1	0.3	0.5	0.7	1.0	1.9	2.8	**3.4**	3.5	3.8	5.3	7.7	8.6	8.7	11.1

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The median is estimated as the middle observed value. Therefore with a probability of about 50% we will be waiting for the bus for less than 3.4 minutes and also for more than 3.4 minutes.

Point estimators

From the measured data we can estimate the real value of the parameter θ using a **point estimator**:

Definition

A **point estimator** of a parameter θ is a function $\hat{\theta}_n(X_1, \ldots, X_n)$ of the random sample which does not depend on θ.

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Notes:

- A point estimator is an example of a statistic. A **statistic** is an arbitrary function of the random sample which does not depend on the parameter θ.
- Generally, we can also construct a point estimator of a function of a parameter $g(\theta)$.
- A typical example is $g(\lambda) = \dfrac{1}{\lambda} = \mathrm{E}\, X$ for the exponential distribution.

Most common point estimators

- **Sample mean** – point estimator of the expectation $\mathrm{E}\, X$:

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- **Sample correlation coefficient** – point estimator of the correlation coefficient $\rho(X, Y)$:

$$r_{X,Y} = r = \frac{s_{X,Y}}{s_X s_Y},$$

where s_X and s_Y are square roots of the sample variances of X and Y.

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Unbiasedness means that an estimator does not have a systematic error, e.g., that it does not produce systematically larger or smaller values.

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The next desirable property of estimators is **consistency**:

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In other words, if for all $\varepsilon > 0$ we have $\mathrm{P}(|\hat{\theta}_n(X_1, \ldots, X_n) - \theta| \geq \varepsilon) \to 0$. Consistency means that by choosing a large n, the error of the estimate will be sufficiently small.

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Theorem

Let $\mathrm{E}\,\hat{\theta}_n^2 < +\infty$ for all n. If for $n \to +\infty$ it holds that

$$\mathrm{E}\,\hat{\theta}_n \to \theta \quad \textit{and} \quad \mathrm{var}\,\hat{\theta}_n \to 0,$$

*then $\hat{\theta}_n$ is a **consistent** estimator.*

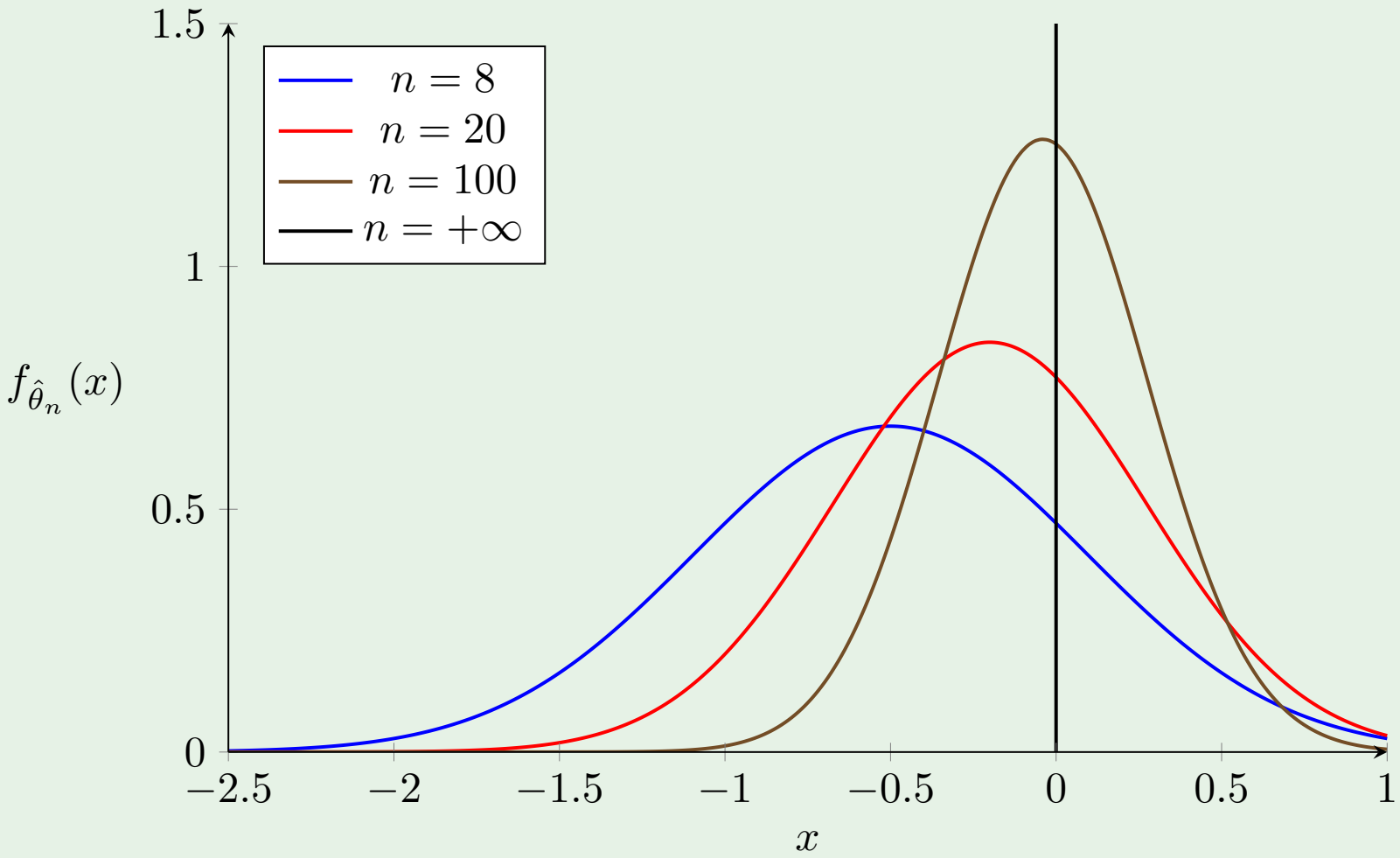
Proof

Proof can be found in bibliography. □

Estimator consistency

Example

Convergence of the densities of a consistent estimator $\hat{\theta}_n$ with the true value of $\theta = 0$.



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- It is also **consistent**: from the weak law of large numbers we get that

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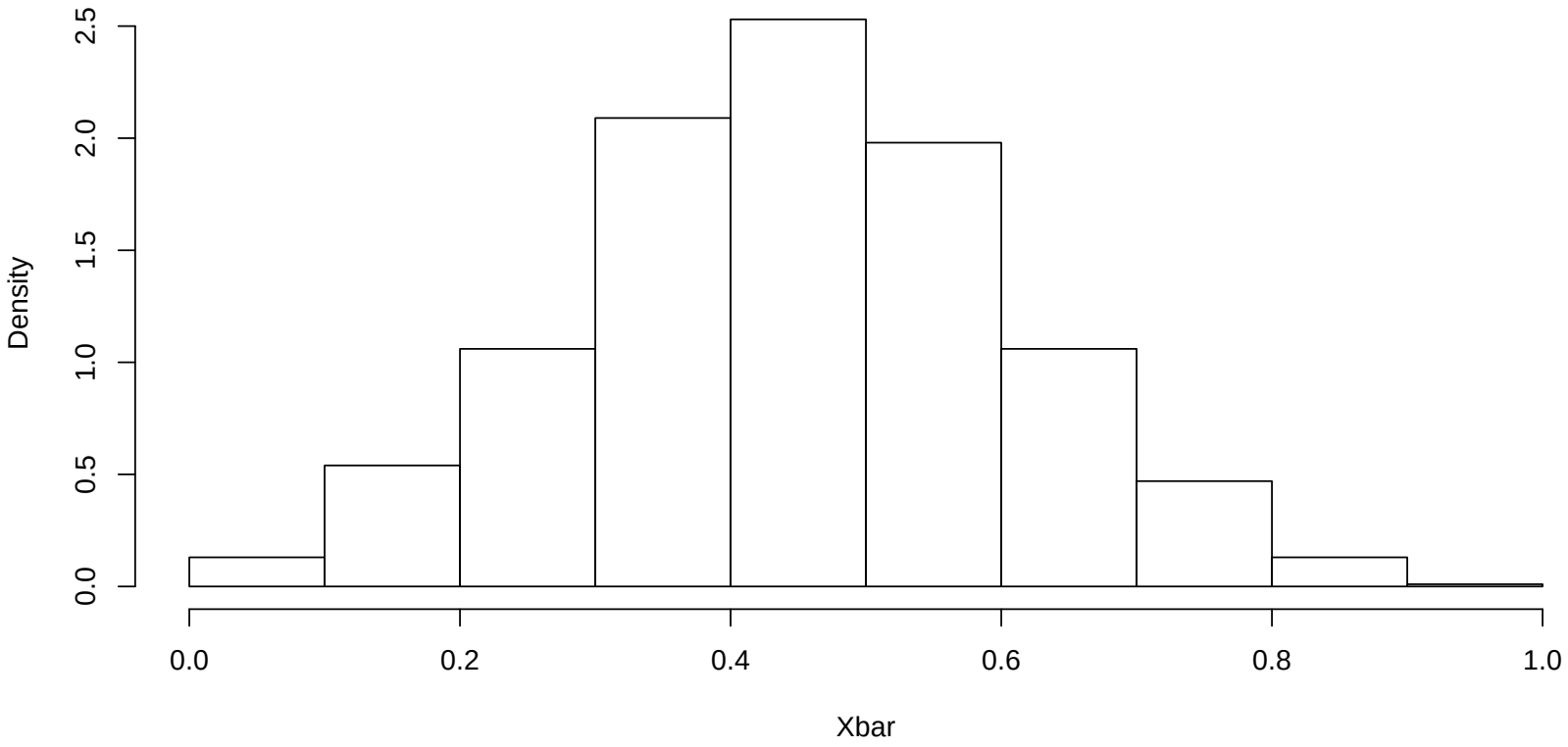
The sample mean $\bar{X}_n$ is thus an unbiased and consistent estimator of the expectation.

Distribution of the sample mean

Histogram of the proportion of heads among 10 coin tosses (1000 simulations).



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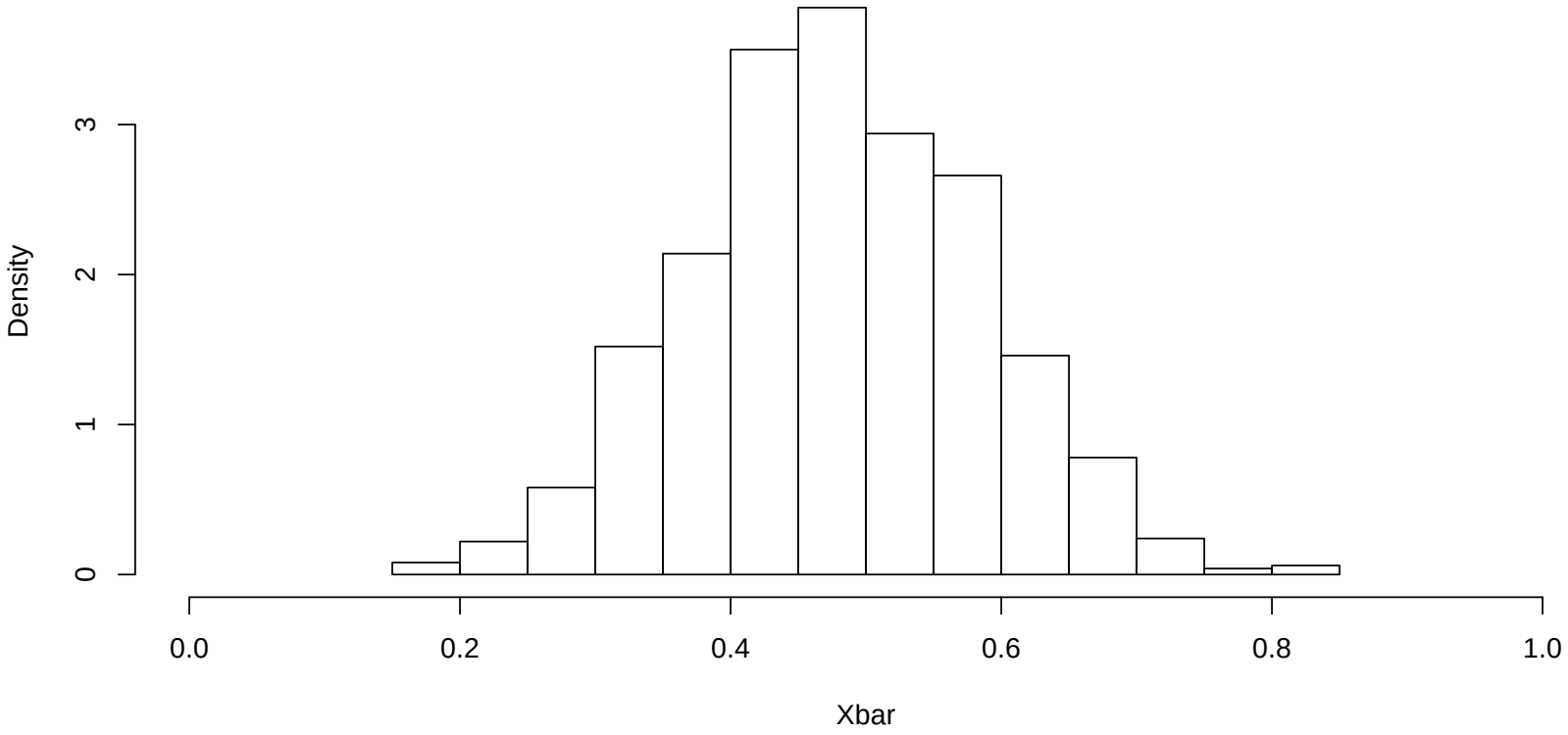
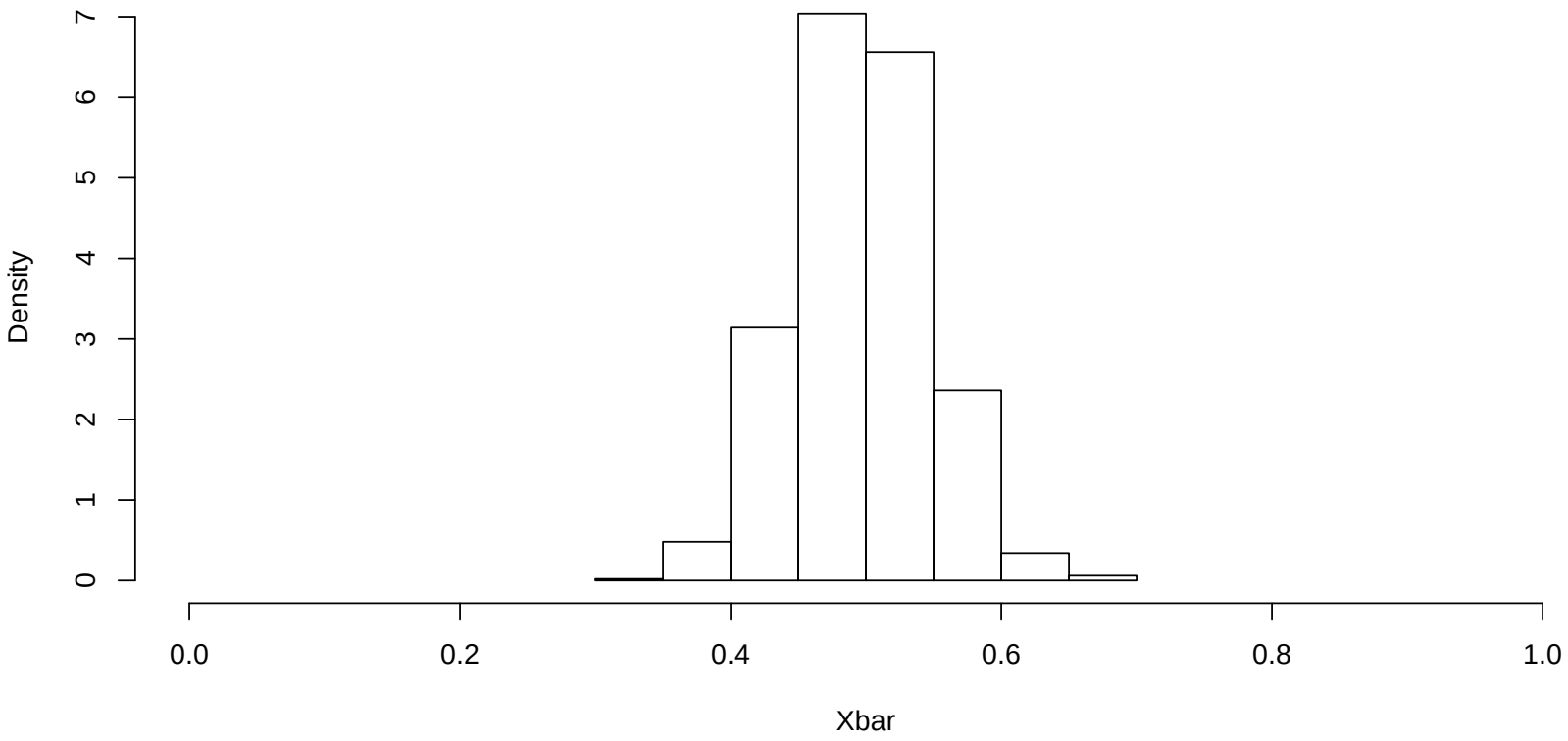


Distribution of the sample mean

Histogram of the proportion of heads among 100 coin tosses (1000 simulations).



Sample variance

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We want to estimate the variance σ^2 using the sample variance s_n^2. First we rewrite s_n^2 as

$$
\begin{aligned}
s_n^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \\
&= \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i^2 - 2 X_i \bar{X}_n + \bar{X}_n^2 \right) \\
&= \frac{1}{n-1} \left(\sum_{i=1}^{n} X_i^2 - 2 \sum_{i=1}^{n} X_i \bar{X}_n + n \bar{X}_n^2 \right) \\
&= \frac{1}{n-1} \left(\sum_{i=1}^{n} X_i^2 - n \bar{X}_n^2 \right) .
\end{aligned}
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- **Unbiasedness**: since $\mathrm{E}\, X_i^2 = \sigma^2 + \mu^2$ and $\mathrm{E}\, \bar{X}_n^2 = \dfrac{\sigma^2}{n} + \mu^2$, we get

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The sample variance s_n^2 is thus an unbiased and consistent estimator of the variance σ^2.

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Theorem

For binomial, Poisson, exponential, and normal distribution the sample mean is the best unbiased estimator of the expected value.
For the normal distribution the sample variance is the best unbiased estimator of the variance.

Method of moments

For a simple and quick (but sometimes not optimal) estimate of the parameters, the **method of moments** can be used. Let $X_1, \ldots, X_n$ be a sample from a distribution with a d-dimensional parameter $\theta = (\theta_1, \ldots, \theta_d)$.

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The method is useful because the law of large numbers implies that $m_k \to \mathrm{E}\, X_i^k$ for $n \to +\infty$. The estimates are thus always consistent.

Methods of moments – estimator of the variance

Suppose $X_1, \ldots, X_n$ form a random sample from a distribution $F_{(\mu,\sigma^2)}$ where $\mathrm{E}\, X_i = \mu$ and $\mathrm{var}\, X_i = \sigma^2$.

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This estimator of the variance is consistent, but not unbiased. However, the extent of the bias will decrease, as $\frac{n-1}{n} \to 1$ for $n \to \infty$.

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Stationary points are 0 and $\frac{3}{4}$ and the maximum is achieved at point $\frac{3}{4}$. Hence we obtain the estimate $\widehat{p}_n = \frac{3}{4}$, which can be guessed from the set up.

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Let the random sample $X_1, \ldots, X_n$ have a distribution given by the joint density

$$f_\theta(\mathbf{x}) = \prod_{i=1}^{n} f_\theta(x_i) \quad \text{for a continuous distribution or}$$

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With values of $\mathbf{x} = (x_1, \ldots, x_n)$ fixed, the function $f_\theta(\mathbf{x})$, or $p_\theta(\mathbf{x})$, as a function of θ is called the **likelihood function** and is denoted as $L(\theta; \mathbf{x})$ or simply $L(\theta)$.

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The likelihood function depends only on the parameter θ. The values $x_1, \ldots, x_n$ are treated as known and fixed.

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Definition

The value $\hat{\theta}_n$ of the parameter θ maximizing the likelihood function $L(\theta; \mathbf{x})$ for a given random sample realization $\mathbf{X} = \mathbf{x}$ is called the **maximum likelihood estimator** (MLE) of the parameter θ. It means that

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- If certain *regularity conditions* are met (see literature), the maximum likelihood estimates are consistent, asymptotically unbiased and asymptotically normal.

Maximum likelihood method – continuous example

Example – parameter of the exponential distribution

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Using the second derivative we can check that the obtained point is indeed the maximum.

Estimating the distribution – example

Example – waiting for a bus – comparison of distributions

Try fitting known continuous distributions on the observed waiting times from before.

Estimate their parameters and compare the densities with the histogram.

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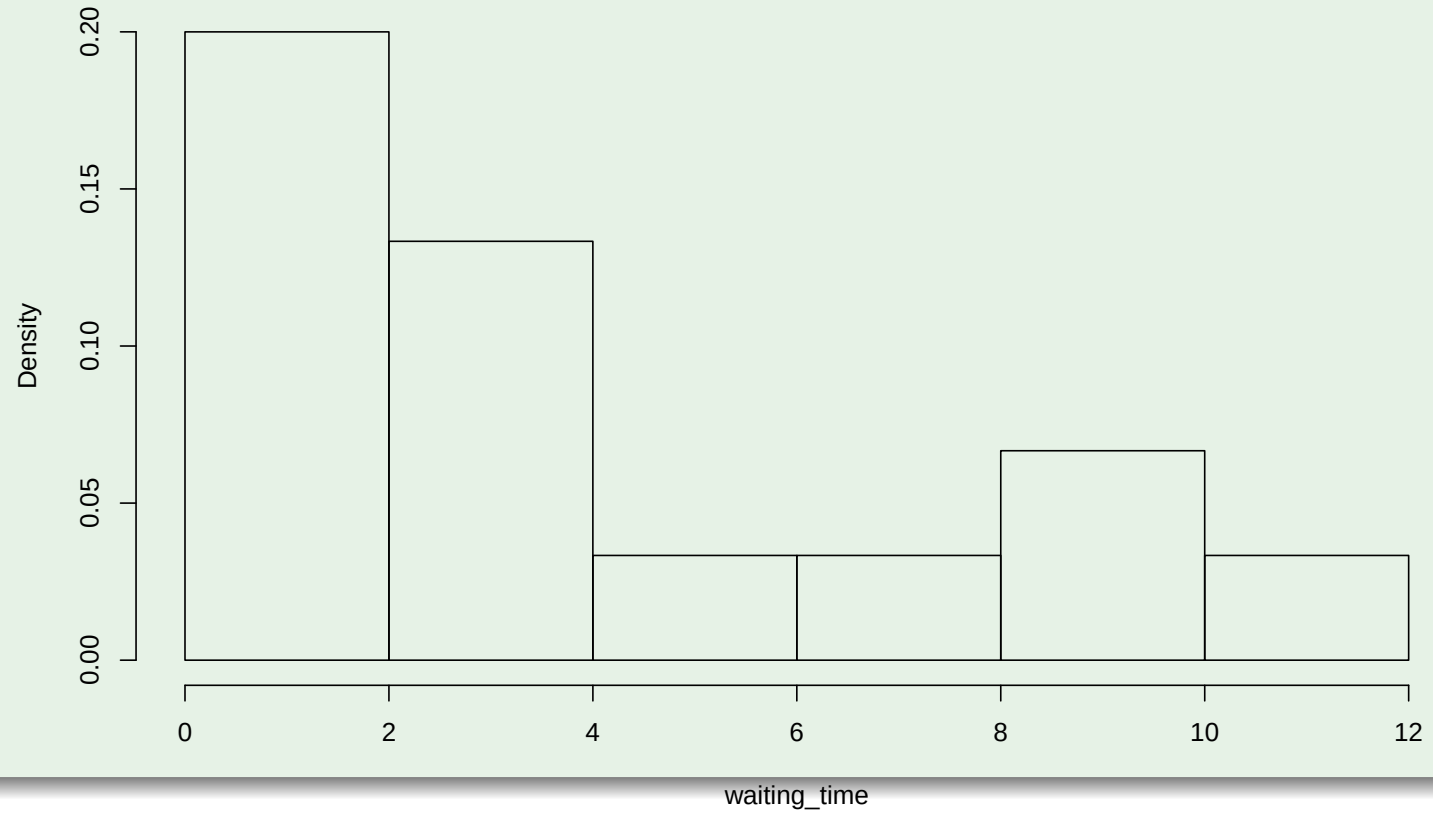
Distribution	Estimated parameters	
Uniform	$a = 0$	$\hat{b}_n = \max(x_1, \ldots, x_{15}) \doteq 11.1$
Exponential	$\hat{\lambda}_n = \frac{1}{\bar{x}_n} \doteq 0.25$	–
Normal	$\hat{\mu}_n = \bar{x}_n \doteq 3.96$	$s_n^2 \doteq 12.56.$

Estimating the distribution – example

Example – waiting for a bus – comparison of distributions, continued

Compare the histogram with the fitted densities.



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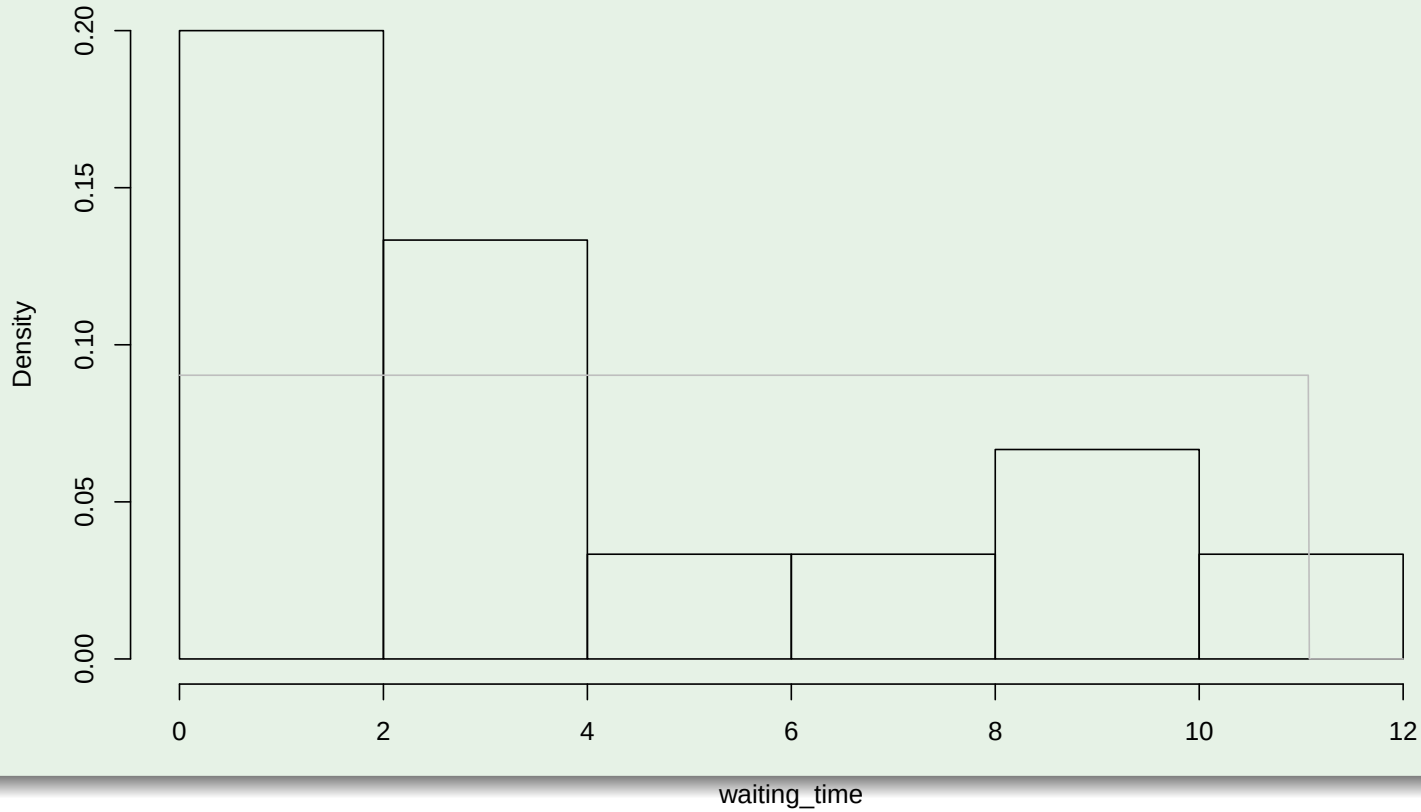

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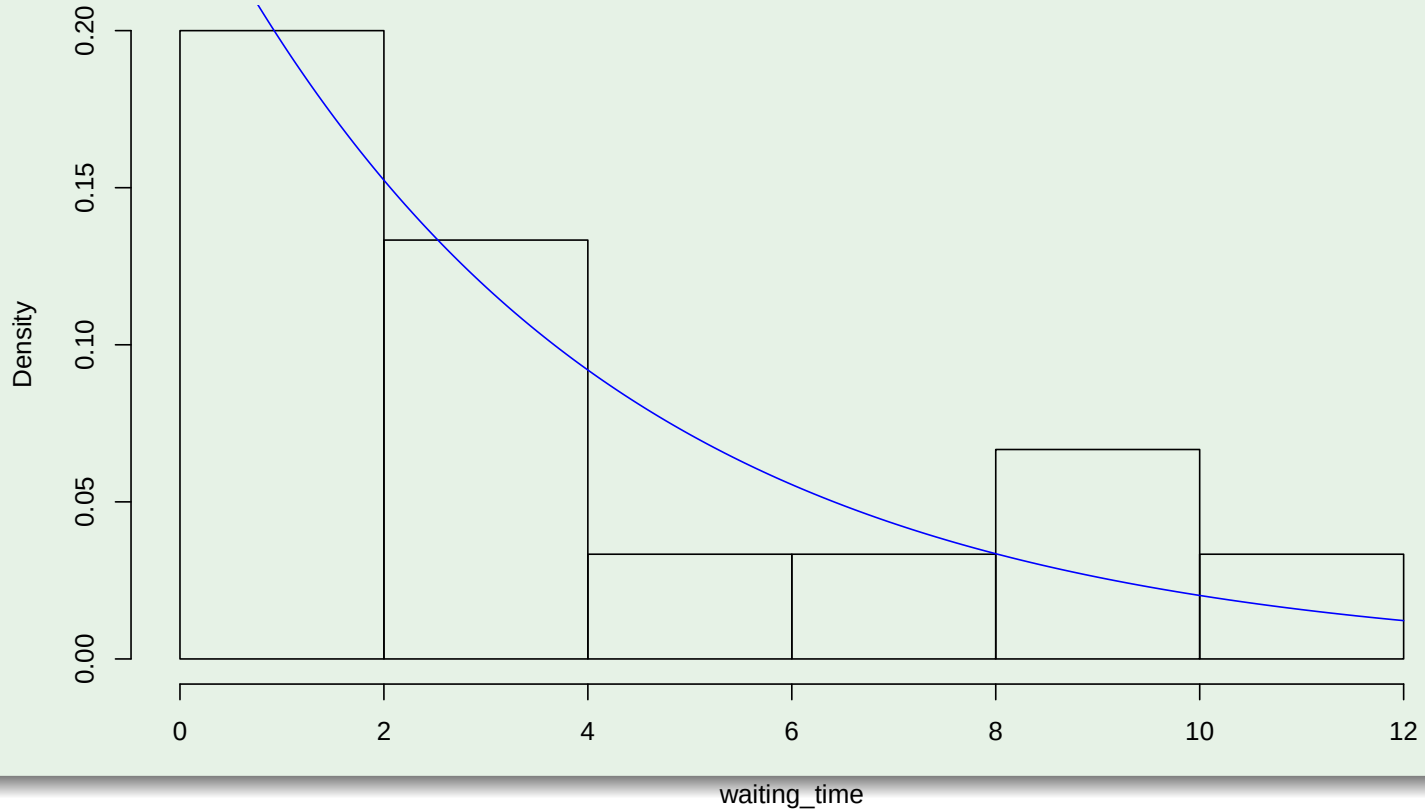

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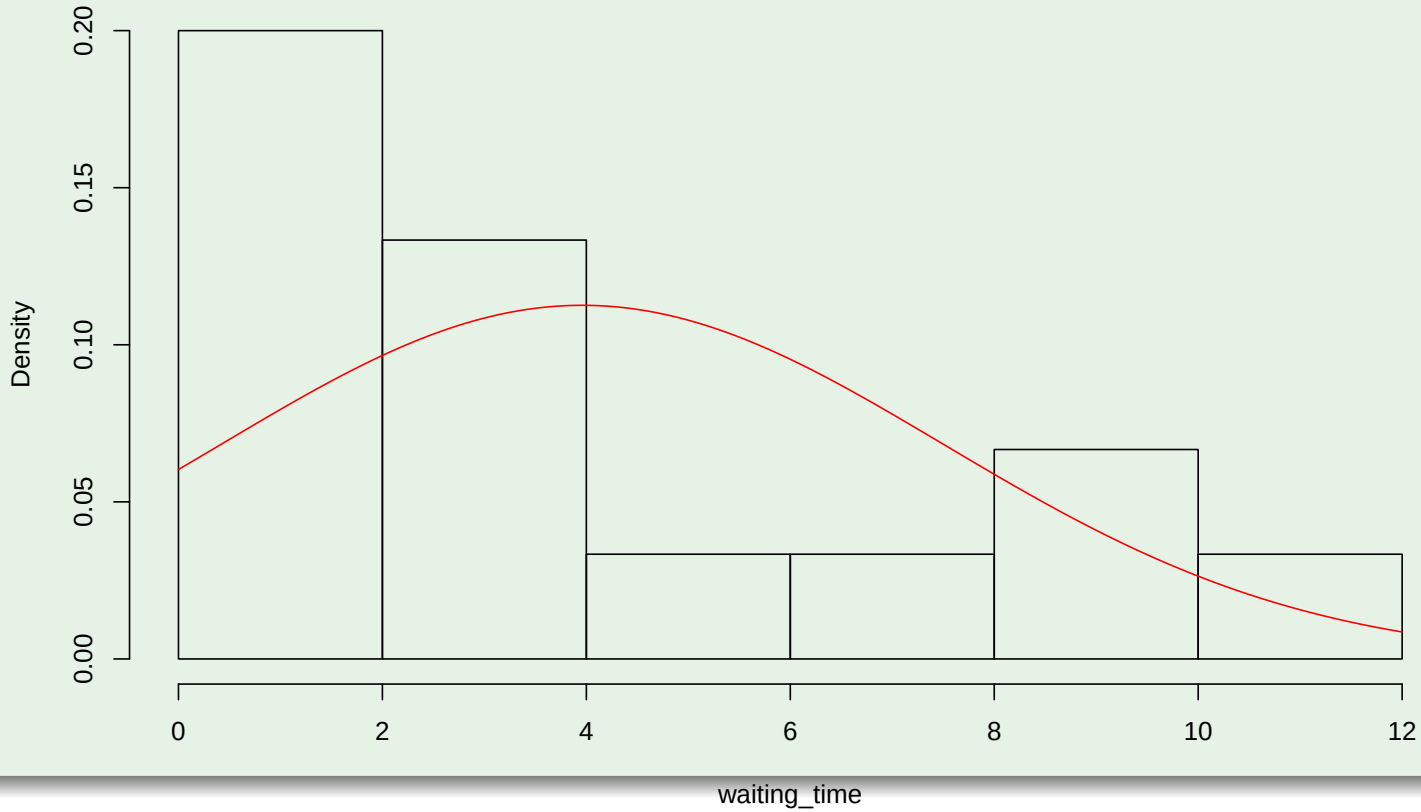

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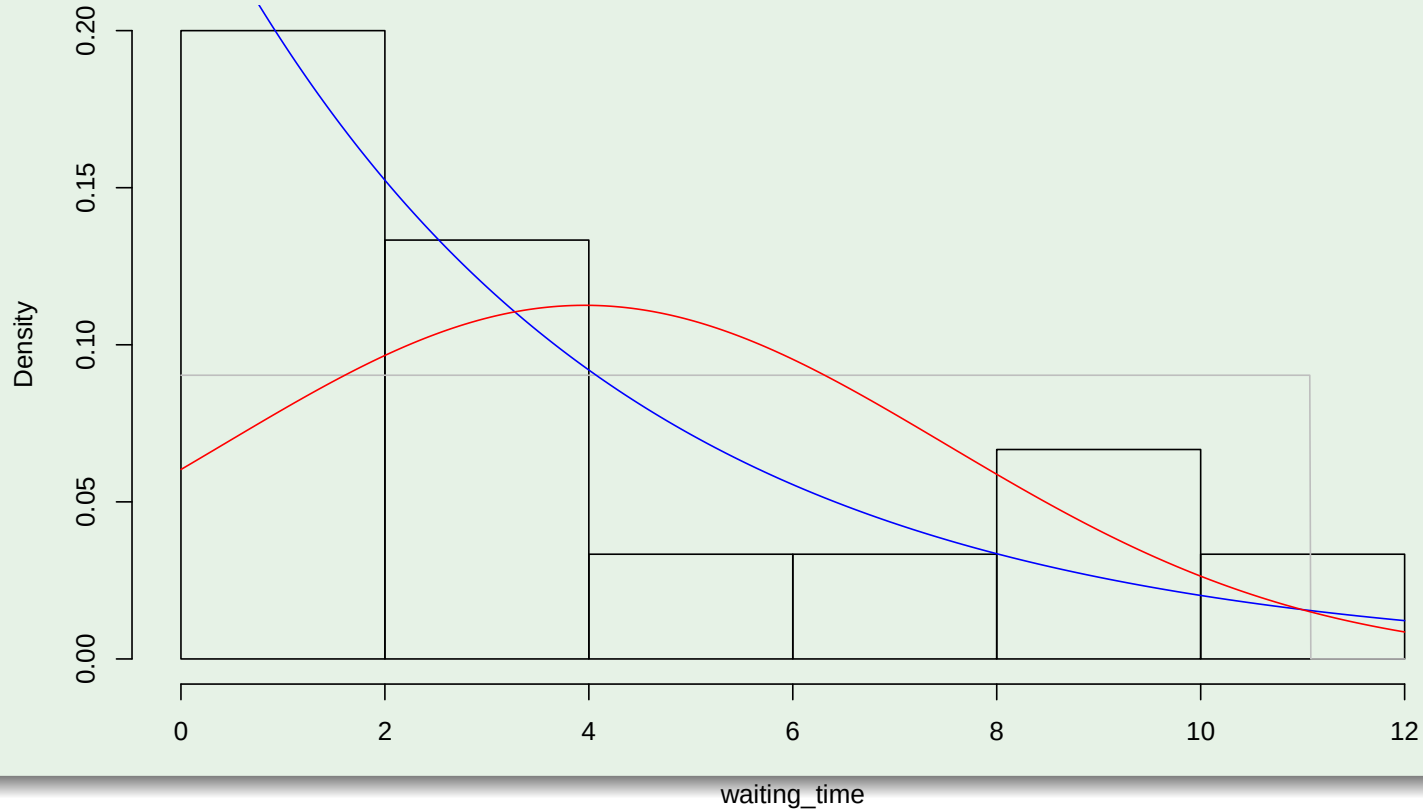

The exponential distribution seems to provide the best fit.

Recap

Suppose we observe a **random sample** $X_1, ..., X_n$ (independent and identically distributed random variables) from an **unknown distribution**. We aim to estimate:

- the **shape** of the distribution – its type and parametric family;
- the **parameters** of the distribution.

To get a graphical overview of the shape of the distribution, we can find:

- The **histogram**, which is an approximation of the **density**.
- The **empirical distribution function**, which estimates the real **distribution function**.

To estimate the parameters θ we use **point estimators** $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$. We want them to be:

- **unbiased**, meaning that $\mathrm{E}\,\hat{\theta}_n = \theta$;
- **consistent**, meaning that $\hat{\theta}_n \xrightarrow{n\to\infty} \theta$.

Estimates with reasonable properties may be found using:

- the **method of moments**;
- the **maximum likelihood method** (MLE).