Conditional probability and independence

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, covariance and correlation, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

A random experiment is represented using a probability space $(\Omega, \mathcal{F}, \mathrm{P})$:

- Ω is the set of possible results;
- $\mathcal{F}$ is a system of subsets of Ω;
- elements $A \in \mathcal{F}$ are called random events;
- the probability measure P is a function, which assigns to the random events a real value from 0 to 1, representing the ideal proportion of cases, in which the events occur.

If there is only a finite many possible results with equal probabilities, then

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|}.$$

Conditional probability

How does the probability change if we have partial information about the result of the experiment?

Example

When rolling a balanced die with no additional information, we know that $\mathrm{P}(4) = 1/6$.

If we know that an even number was rolled, then it is clear that $\mathrm{P}(4 \,|\, \text{even}) = 1/3$.

Conditional probability

Consider the uniform distribution on a set Ω with a finite “size” (e.g., the number of elements, length, area, capacity, time, etc.).
The probability of an event A is then defined as the by ratio of “sizes” as

$$\mathrm{P}(A) = \text{size}(A)/\text{size}(\Omega).$$

If we know that an event B surely occurred, we are in fact interested only in outcomes of the experiment favorable to the event B. Favorable outcomes to the event A are now in $A \cap B$ and all of them must be in B (B surely occurred). We have

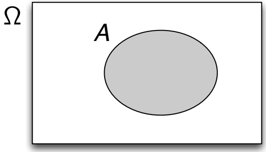
$$\mathrm{P}(A|B) = \frac{\text{size}(A \cap B)}{\text{size}(B)} = \frac{\text{size}(A \cap B)/\text{size}(\Omega)}{\text{size}(B)/\text{size}(\Omega)} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Definition

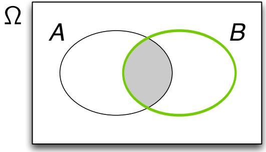
Let A, B be events and $\mathrm{P}(B) > 0$. The **conditional probability** of the event A given (the event) B is denoted by $\mathrm{P}(A|B)$ and is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Conditional probability – Venn diagram



$$P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}$$



$$P(A \text{ given } B) = \frac{\text{area}(\text{part of } A \text{ inside } B)}{\text{area}(B)}$$

$$P(A|B) = \frac{\text{area}(A \cap B) \; / \; \text{area}(\Omega)}{\text{area}(B) \quad / \; \text{area}(\Omega)}$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$P(A \cap B) = P(A|B) \; P(B)$$

$$P(A \cap B) = P(B|A) \; P(A)$$

$$P(\text{intersection}) = P(\text{event} \mid \text{condition}) \; P(\text{condition})$$

Conditional probability – examples

Example – rolling two dice

Consider two rolls of a die. What is $\mathrm{P}(\text{sum} > 6 \mid \text{first} = 3)$?

The answer is surely $1/2$, since the second rolled number must be $4, 5$, or 6.

Formally: $\Omega = \{1, 2, 3, 4, 5, 6\}^2$,
$\mathrm{P}(A) = |A|/36$ for each $A \subset \Omega$.
Let $B = \{(3, \omega_2) : 1 \leq \omega_2 \leq 6\}, \;\; A = \{(\omega_1, \omega_2) : \omega_1 + \omega_2 > 6\}$.
Then

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\frac{|A \cap B|}{36}}{\frac{|B|}{36}} = \frac{|A \cap B|}{|B|} = \frac{3}{6}.$$

Conditional probability – examples

Example – family with two children

A trickier example:

A family has two children. What is the probability that both are boys, given that at least one of them is a boy? I.e., what is the value of $\mathrm{P}(\text{both boys} \mid \text{at least one is a boy})$?

$\Omega = \{GG, GB, BG, BB\}$.

$$\mathrm{P}(BB|BG \cup GB \cup BB) = \frac{\mathrm{P}(BB \cap (BG \cup GB \cup BB))}{\mathrm{P}(BG \cup BG \cup BB)} = \frac{\mathrm{P}(BB)}{\mathrm{P}(BG \cup GB \cup BB)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

<u>Incorrect:</u> $\mathrm{P}(BB|\text{older is boy}) = \mathrm{P}(BB|BG \cup BB) = \dfrac{\mathrm{P}(BB \cap (BG \cup BB))}{\mathrm{P}(BG \cup BB)} = \dfrac{1}{2}$.

Properties of conditional probability

Lemma

Let $\mathrm{P}(B) > 0$*. Then the conditional probability* $\mathrm{P}(\cdot|B)$ *is a probability measure, i.e.,* $\mathrm{P}(\cdot|B) \in [0, 1]$ *and it fulfills the axioms of probability.*

Proof

We need to prove the following:

- **i)** $\mathrm{P}(\cdot|B) : \mathcal{F} \to \mathbb{R}$,
- **ii)** **non-negativity:** for all $A \in \mathcal{F}$ it holds $\mathrm{P}(A|B) \geq 0$,
- **iii)** **normalization:** $\mathrm{P}(\Omega|B) = 1$, $\qquad \mathrm{P}(\Omega|B) = \dfrac{\mathrm{P}(\Omega \cap B)}{\mathrm{P}(B)} = \dfrac{\mathrm{P}(B)}{\mathrm{P}(B)} = 1$,
- **iv)** $\sigma-$**additivity**: If $A_1, A_2, \ldots \in \mathcal{F}$ are mutually disjoint events (i.e., $A_i \cap A_j = \emptyset$ for $\forall\, i, j : i \neq j$), then

$$\mathrm{P}\left(\bigcup_{i=1}^{+\infty} A_i \middle| B\right) = \frac{\mathrm{P}\left(\left(\bigcup_{i=1}^{+\infty} A_i\right) \cap B\right)}{\mathrm{P}(B)} = \frac{\mathrm{P}\left(\bigcup_{i=1}^{+\infty} (A_i \cap B)\right)}{\mathrm{P}(B)} = \cdots = \sum_{i=1}^{+\infty} \mathrm{P}(A_i|B).$$

□

Properties of conditional probability

Conditional probability fulfills all mentioned properties of probability as well:

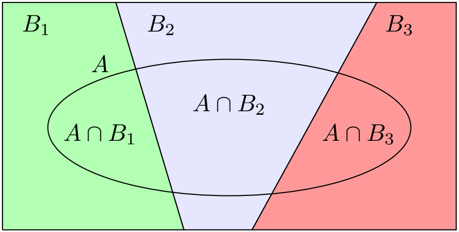
- if A_1 and A_2 are mutually disjoint, then $\mathrm{P}(A_1 \cup A_2|B) = \mathrm{P}(A_1|B) + \mathrm{P}(A_2|B)$,
- $\mathrm{P}(A_1 \cup A_2|B) = \mathrm{P}(A_1|B) + \mathrm{P}(A_2|B) - \mathrm{P}(A_1 \cap A_2|B)$,
- $\mathrm{P}(A^c|B) = 1 - \mathrm{P}(A|B)$,
- etc.

Moreover, the probability $\mathrm{P}(A|B)$ “lives” on B: for $A \cap B = \emptyset$ we have $\mathrm{P}(A|B) = 0$.

Furthermore, $\mathrm{P}(A \cap B|B) = \dfrac{\mathrm{P}(A \cap B \cap B)}{\mathrm{P}(B)} = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A|B)$.

Case distinct formula (Law of total probability)

$\Omega = B_1 \cup B_2 \cup B_3$ (disjoint partition)



Recall:

$$\mathrm{P}(A|B_i) = \frac{\mathrm{P}(A \cap B_i)}{\mathrm{P}(B_i)}$$

$$\mathrm{P}(A \cap B_i) = \mathrm{P}(A|B_i)\,\mathrm{P}(B_i)$$

$$A = A \cap \Omega = A \cap (B_1 \cup B_2 \cup B_3)$$

$$= (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$$

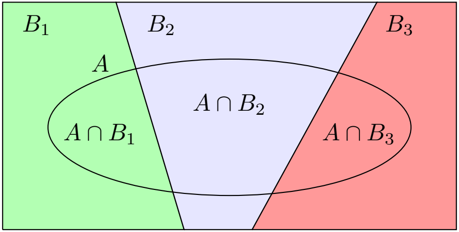
$$\mathrm{P}(A) = \mathrm{P}(A \cap B_1) + \mathrm{P}(A \cap B_2) + \mathrm{P}(A \cap B_3)$$

$$\mathrm{P}(A) = \mathrm{P}(A|B_1)\,\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\,\mathrm{P}(B_2) + \mathrm{P}(A|B_3)\,\mathrm{P}(B_3)$$

Bayes' Theorem = converse procedure

At the end we observe A and we ask ourselves, what is the probability that the event B_j occurred.

$\Omega = B_1 \cup B_2 \cup B_3$ (disjoint partition)



Recall:

$$\mathrm{P}(A \cap B_j) = \mathrm{P}(A|B_j)\,\mathrm{P}(B_j)$$

$$\mathrm{P}(A) = \mathrm{P}(A|B_1)\,\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\,\mathrm{P}(B_2) + \mathrm{P}(A|B_3)\,\mathrm{P}(B_3)$$

$$\mathrm{P}(B_j|A) = \frac{\mathrm{P}(A \cap B_j)}{\mathrm{P}(A)}$$

$$\mathrm{P}(B_j|A) = \frac{\mathrm{P}(A|B_j)\,\mathrm{P}(B_j)}{\mathrm{P}(A|B_1)\,\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\,\mathrm{P}(B_2) + \mathrm{P}(A|B_3)\,\mathrm{P}(B_3)}$$

Bayes' Theorem (Thomas Bayes, 1701–1761)

A family of mutual disjoint events $B_1, B_2, \dots B_n$ is called a **partition** of the set Ω, if

$$\Omega = \bigcup_{i=1}^{n} B_i.$$

Theorem – case distinct formula (Law of total probability)

Let $B_1, B_2, \dots, B_n$ be a partition of Ω such that $\forall\, i : \mathrm{P}(B_i) > 0$. Then for each event A it holds that

$$\mathrm{P}(A) = \sum_{i=1}^{n} \mathrm{P}(A|B_i)\,\mathrm{P}(B_i).$$

Theorem – Bayes' Theorem

Let $B_1, B_2, \dots, B_n$ be a partition of Ω such that $\forall\, i : \mathrm{P}(B_i) > 0$ and let A be an event with $\mathrm{P}(A) > 0$. Then it holds that

$$\mathrm{P}(B_j|A) = \frac{\mathrm{P}(A|B_j)\,\mathrm{P}(B_j)}{\sum_{i=1}^{n} \mathrm{P}(A|B_i)\,\mathrm{P}(B_i)}.$$

Bayes' Theorem – example

Example – spam filter

From the analysis of our email account we find out that:

- 30% of all delivered messages is spam;
- in 70% of spam messages there is the word "copy";
- only in 10% of non-spam messages there is the word "copy".

Calculate the probability that a message containing the word "copy" is a spam,

S: set of spam messages,
$S^c = \Omega \setminus S$: set of non-spam messages,
C: set of messages containing word "copy",
C^c: set of messages not containing the word "copy".

$$\mathrm{P}(S) = 0.3,\ \mathrm{P}(S^c) = 0.7, \qquad \mathrm{P}(C|S) = 0.7,\ \mathrm{P}(C|S^c) = 0.1$$

$$\mathrm{P}(S|C) = \frac{\mathrm{P}(C|S)\,\mathrm{P}(S)}{\mathrm{P}(C|S)\,\mathrm{P}(S) + \mathrm{P}(C|S^c)\,\mathrm{P}(S^c)} = \frac{0.7 \cdot 0.3}{0.7 \cdot 0.3 + 0.1 \cdot 0.7} = \frac{21}{28} = 0.75.$$

Probability trees

First let us recall a useful relation:

From the definition of conditional probability it follows that:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\,\mathrm{P}(B).$$

For 3 events it similarly holds that:

$$\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A)\,\mathrm{P}(B|A)\,\mathrm{P}(C|A \cap B),$$

which can be proven by using the definition of conditional probability on the right hand side:

$$\begin{aligned}\mathrm{P}(A)\,\mathrm{P}(B|A)\,\mathrm{P}(C|A \cap B) &= \mathrm{P}(A)\frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}\frac{\mathrm{P}(C \cap (A \cap B))}{\mathrm{P}(A \cap B)} \\ &= \mathrm{P}(A \cap B \cap C).\end{aligned}$$

Probability trees

Lemma – Multiplicative law

Let for events $A_1, \ldots, A_n$ *hold that* $\mathrm{P}(A_1 \cap \cdots \cap A_n) > 0$. *Then it holds that*

$$\mathrm{P}(A_1 \cap \cdots \cap A_n) = \mathrm{P}(A_1)\,\mathrm{P}(A_2|A_1)\,\mathrm{P}(A_3|A_1 \cap A_2) \ldots \mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

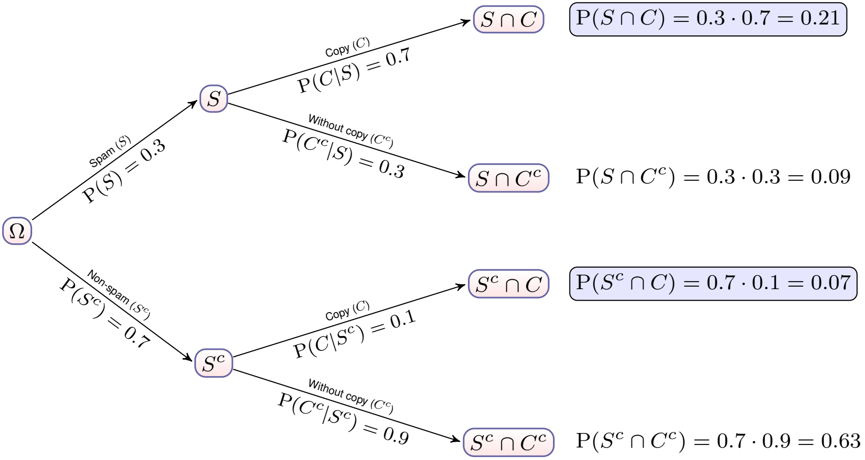
Proof

We apply successively the relation $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B|A)$ following from the definition of conditional probability:

$$\begin{aligned}
\mathrm{P}(A_1 \cap \cdots \cap A_n) &= \mathrm{P}(A_1 \cap \cdots \cap A_{n-1})\,\mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1}) \\
&= \mathrm{P}(A_1 \cap \cdots \cap A_{n-2})\,\mathrm{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2})\,\mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1}) \\
&= \ldots .
\end{aligned}$$

□

Example – spam filter



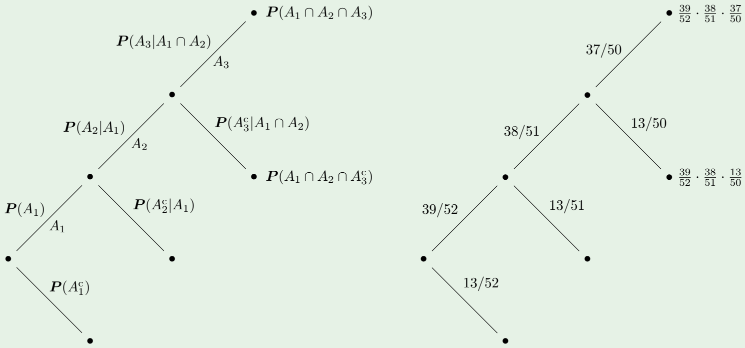
$$P(S|C) = \frac{P(S \cap C)}{P(C)} = \frac{0.21}{0.21 + 0.07} = 0.75$$

Example – multiplicative law

Suppose we draw cards without replacement from a 52 cards deck. What is the probability that in a sequence of 3 cards drawn one after another there are no hearts?

$A_i = \{i\text{-th card is not hearts}\}, \quad i = 1, 2, 3.$

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\,\mathrm{P}(A_2|A_1)\,\mathrm{P}(A_3|A_1 \cap A_2) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \doteq 41.4\%.$$



The probability of a given vertex of the tree is the product of the corresponding values on the path stemming from the root.

Misinterpretations of conditional probability

Many data misinterpretations and fallacies are based on incorrect understanding of conditional probabilities:

Example – driving under influence

- It was observed that approximatively 10% of fatal car accidents are caused by drunk drivers (46 out of 454 road fatalities in 2022 in the Czech Republic according to the yearly police report).
- This means that 90% of fatal accidents are caused by sober drivers!
- Does this mean that we should should beware of the sober drivers?

Misinterpretations of conditional probability

Example – driving under influence continued

Does this mean that we should should beware of the sober drivers?

Of course not. We have to carefully read the probabilities.

The figure tells us that among all accidents, the percentage caused by drunk drivers is 10%. Thus

$$\mathrm{P}(\text{drunk}|\text{accident}) = 0.1.$$

What we are trying to find out is the reverse conditional probability $\mathrm{P}(\text{accident}|\text{drunk})$.

From a different study, we have found out that less than 1% of drivers are driving under influence. The overall chance of accident is difficult to determine, so we will compute just how more likely it is to cause an accident for drunk drivers:

$$\begin{aligned}\frac{\mathrm{P}(\text{accident}|\text{drunk})}{\mathrm{P}(\text{accident}|\text{sober})} &= \frac{\mathrm{P}(\text{accident} \cap \text{drunk}) / \mathrm{P}(\text{drunk})}{\mathrm{P}(\text{accident} \cap \text{sober}) / \mathrm{P}(\text{sober})} \\ &= \frac{\mathrm{P}(\text{drunk}|\text{accident}) \cdot \mathrm{P}(\text{accident}) / \mathrm{P}(\text{drunk})}{\mathrm{P}(\text{sober}|\text{accident}) \cdot \mathrm{P}(\text{accident}) / \mathrm{P}(\text{sober})} = \frac{0.1/0.01}{0.9/0.99} = 11.\end{aligned}$$

Drunk drivers have at least 11 times higher probability of causing a fatal accident.

Independence of events

Intuitively: A and B are independent if the probability of the event A is not influenced by the knowledge about occurrence of the event B, i.e., $\mathrm{P}(A|B) = \mathrm{P}(A)$, and (vice versa) $\mathrm{P}(B|A) = \mathrm{P}(B)$.

Definition

Events A and B are called **independent**, if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B).$$

Generally, a family of events $\{A_i \,|\, i \in I\}$ is called **independent** if

$$\mathrm{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathrm{P}(A_i)$$

for all finite non-empty subsets J of I.

Independence of events

Example – rolling a die

Consider the events

A: "an even number is rolled" and B: "a number less than 3 is rolled".

Are the events A and B independent?

$$A = \{2, 4, 6\}, \qquad B = \{1, 2\}, \qquad A \cap B = \{2\}.$$

$$\mathrm{P}(A \cap B) = \frac{1}{6} \qquad \text{and} \qquad \mathrm{P}(A)\,\mathrm{P}(B) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}.$$

Then the events A and B are independent.

Independence of events

Example – rolling a die

Consider the events

A: "an even number is rolled" and B: "number 4 is rolled".

Are the events A and B independent?

$$A = \{2, 4, 6\}, \qquad B = \{4\}, \qquad A \cap B = \{4\}.$$

$$\mathrm{P}(A \cap B) = \frac{1}{6} \qquad \text{and} \qquad \mathrm{P}(A)\,\mathrm{P}(B) = \frac{3}{6} \cdot \frac{1}{6} = \frac{1}{12}.$$

Then events A and B are not independent.

Relation between independence and conditional probability

Let A and B be independent events and $\mathrm{P}(B) > 0$. Then clearly

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A)\,\mathrm{P}(B)}{\mathrm{P}(B)} = \mathrm{P}(A).$$

For A and B independent the knowledge of B does not bring us any information about A.

Theorem

If the events A and B are independent then A and B^c (resp., A^c and B; A^c and B^c) are independent, too.

Theorem

If $(A_i)_{i\in I}$ is a family of independent events, then for any arbitrary non-empty finite subset $\emptyset \neq J \subset I$ it holds that

$$\mathrm{P}\left(\bigcap_{i\in J} A_i \mid \bigcap_{i\in I\setminus J} A_i\right) = \mathrm{P}\left(\bigcap_{i\in J} A_i\right).$$

Independent vs disjoint events

A common error is to make the fallacious statement that A and B are independent if $A \cap B = \emptyset$.

In fact, disjoint events A and B are independent only if $\mathrm{P}(A) = 0$ or $\mathrm{P}(B) = 0$.

If A and B are disjoint with non-zero probabilities, then the knowledge that B occurred tells us that A cannot occur.

The events being disjoint is a matter of sets, independence is a matter of probabilities.

Conditional independence

Definition

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and C an event with $\mathrm{P}(C) > 0$. Events A and B are called **conditionally independent with respect to C**, if

$$\mathrm{P}(A \cap B | C) = \mathrm{P}(A|C)\,\mathrm{P}(B|C).$$

Recall:

- $Q(A) = \mathrm{P}(A|C)$ is a probability measure;
- the conditional independence is thus the independence with respect to probability Q.

Conditional independence

Example – rolling a seven-sided die

Suppose we roll a seven-sided die with all sides equally likely. Consider the events:

A: "an even number is rolled", B: "a number less than 3 is rolled".

Are the events A and B independent? $A = \{2, 4, 6\},\ B = \{1, 2\}, A \cap B = \{2\}$.

$$\mathrm{P}(A \cap B) = \frac{1}{7} \quad \text{and} \quad \mathrm{P}(A) \cdot \mathrm{P}(B) = \frac{3}{7} \cdot \frac{2}{7} = \frac{6}{49}.$$

Events A and B are not independent.

Example – rolling a seven-sided die + condition

Consider further event C: "we rolled at most 6" $C = \{1, 2, 3, 4, 5, 6\}$.

Are events A and B conditionally independent with respect to C?

$$\mathrm{P}(A \cap B | C) = \frac{\mathrm{P}(A \cap B \cap C)}{\mathrm{P}(C)} = \frac{\mathrm{P}(\{2\})}{\mathrm{P}(\{1, \ldots, 6\})} = \frac{1/7}{6/7} = \frac{1}{6},$$

$$\mathrm{P}(A|C) \cdot \mathrm{P}(B|C) = \frac{3/7}{6/7} \cdot \frac{2/7}{6/7} = \frac{1}{6}.$$

Events A and B are conditionally independent with respect to C.

Recap

- The **conditional probability** that an event A occurs if we know that an event B with $\mathrm{P}(B) > 0$ occured, is defined as $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.
- **Law of total probability**: For A and B with $\mathrm{P}(B) > 0$ we have

$$\mathrm{P}(A) = \mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B^c)\,\mathrm{P}(B^c).$$

- **Bayes' Theorem**: For A and B with $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$ we have

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A|B)\,\mathrm{P}(B)}{\mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B^c)\,\mathrm{P}(B^c)}.$$

- Events A and B are called **independent** if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B).$$

- For **independent** events A and B the knowledge that one of them occurred or not does not change the probability of the other one happening:

$$\mathrm{P}(A|B) = \mathrm{P}(A) \qquad \text{and} \qquad \mathrm{P}(B|A) = \mathrm{P}(B).$$