Random vectors II.

(Covariance and correlation, convolution)

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Probability and Statistics

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Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' Theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value, covariance and correlation.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

Joint distribution function of a random vector (X, Y):

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x \cap Y \leq y).$$

Discrete random variables X and Y	**Continuous random variables X and Y**
Joint probabilities of values:	Joint density:
$\mathrm{P}(X = x \cap Y = y)$	$f_{X,Y}(x, y)$

Marginal distributions:

$\mathrm{P}(X = x) = \sum_{\text{all } y} \mathrm{P}(X = x \cap Y = y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$
$\mathrm{P}(Y = y) = \sum_{\text{all } x} \mathrm{P}(X = x \cap Y = y)$	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x$

Independence of X and Y:

$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y)$	$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$

Conditional probabilities / density of X given $Y = y$:

$\mathrm{P}(X = x \mid Y = y) = \dfrac{\mathrm{P}(X = x \cap Y = y)}{\mathrm{P}(Y = y)}$	$f_{X\mid Y}(x\mid y) = \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$

Conditional expectation of X given $Y = y$:

$\mathrm{E}(X\mid Y = y) = \sum_{x} x\,\mathrm{P}(X = x\mid Y = y)$	$\mathrm{E}(X\mid Y = y) = \int_{-\infty}^{\infty} x f_{X\mid Y}(x\mid y)\,\mathrm{d}x$

Functions of random vectors of variables

Similar formulas as for a function of one random variable also hold for the functions of random vectors. Let

$$Z = h(X_1, \ldots, X_n) = h(\boldsymbol{X}).$$

- When variables $X_1, \ldots, X_n$ have a joint discrete distribution with probabilities $\mathrm{P}(\boldsymbol{X} = \boldsymbol{x})$, the following relation holds for the distribution function of Z:

$$F_Z(z) = \mathrm{P}(Z \leq z) = \sum_{\{\boldsymbol{x} \in \mathbb{R}^n :\, h(\boldsymbol{x}) \leq z\}} \mathrm{P}(\boldsymbol{X} = \boldsymbol{x}).$$

- When variables $X_1, \ldots, X_n$ have a joint continuous distribution with density $f_{\boldsymbol{X}}(\boldsymbol{x})$, the distribution function of Z is then

$$F_Z(z) = \mathrm{P}(Z \leq z) = \int \cdots \int_{\{\boldsymbol{x} \in \mathbb{R}^n :\, h(\boldsymbol{x}) \leq z\}} f_{\boldsymbol{X}}(\boldsymbol{x}) \,\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Expected value of the function of a random vector

The expected value $\mathrm{E}\, h(X, Y)$ of a real function h of random variables X and Y can be computed without determining the distribution of the variable $h(X, Y)$.

- For X and Y discrete random variables it holds that

$$\mathrm{E}\, h(X, Y) = \sum_{i,j} h(x_i, y_j)\, \mathrm{P}(X = x_i \cap Y = y_j),$$

  if the sum converges absolutely.

- For X and Y continuous random variables it holds that

$$\mathrm{E}\, h(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x, y) f_{X,Y}(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

  if the integral converges absolutely.

Properties of the expected value

Now we can prove the linearity of the expectation.

Theorem – linearity of expectation

For all $a, b \in \mathbb{R}$ and all random variables X and Y it holds that

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}\,X + b\,\mathrm{E}\,Y.$$

Consequence:

- $\mathrm{E}(aX + b) = a\,\mathrm{E}\,X + b$. This statement was proven before separately.

Properties of the expected value

Proof

From the theory concerning the marginal distributions of discrete random variables X and Y we have:

$$\begin{aligned}
\mathrm{E}(aX + bY) &= \sum_{i,j} (ax_i + by_j)\,\mathrm{P}(X = x_i \cap Y = y_j) \\
&= \sum_{i,j} ax_i\,\mathrm{P}(X = x_i \cap Y = y_j) + \sum_{i,j} by_j\,\mathrm{P}(X = x_i \cap Y = y_j) \\
&= a\sum_i x_i \sum_j \mathrm{P}(X = x_i \cap Y = y_j) + b\sum_j y_j \sum_i \mathrm{P}(X = x_i \cap Y = y_j) \\
&= a\sum_i x_i\,\mathrm{P}(X = x_i) + b\sum_j y_j\,\mathrm{P}(Y = y_j) \quad = \quad a\,\mathrm{E}\,X + b\,\mathrm{E}\,Y.
\end{aligned}$$

For continuous X and Y the proof is analogous:

$$\begin{aligned}
\mathrm{E}(aX + bY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax + by) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \cdots = \\
&= a\int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y \quad = \quad a\,\mathrm{E}\,X + b\,\mathrm{E}\,Y.
\end{aligned}$$

□

Covariance and correlation coefficient

Mutual linear dependence of two random variables X and Y can be described in the following way:

Definition

Let X and Y be random variables with finite second moments. Then we define the **covariance** of the random variables X and Y as

$$\operatorname{cov}(X, Y) = \operatorname{E}[(X - \operatorname{E} X)(Y - \operatorname{E} Y)].$$

If X and Y have positive variances then we define the **correlation coefficient** (or **coefficient of correlation**) as

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}}.$$

Definition

Two random variables X and Y are called **non-correlated** if $\operatorname{cov}(X, Y) = 0$.

Covariance and the correlation coefficient – properties

Theorem

For the covariance and the correlation coefficient the following properties hold:

- **i)** $\operatorname{cov}(X,Y) = \mathrm{E}\,XY - \mathrm{E}\,X\,\mathrm{E}\,Y$,
- **ii)** X *and* Y *are non-correlated if and only if* $\mathrm{E}\,XY = \mathrm{E}\,X\,\mathrm{E}\,Y$,
- **iii)** $\rho(X,Y) \in [-1,1]$,
- **iv)** $\rho(aX+b, cY+d) = \rho(X,Y)$ *for all* $a, c > 0$ *and* $b, d \in \mathbb{R}$,
- **v)** $\rho(X,Y) = \pm 1$, *if* $a, b \in \mathbb{R}$, $a > 0$ *such that* $Y = \pm aX + b$.

Proof

i)
$$\begin{aligned}\operatorname{cov}(X,Y) &= \mathrm{E}\left((X-\mathrm{E}\,X)(Y-\mathrm{E}\,Y)\right) = \mathrm{E}\left(XY - X\,\mathrm{E}\,Y - Y\,\mathrm{E}\,X + \mathrm{E}\,X\,\mathrm{E}\,Y\right)\\ &= \mathrm{E}\,XY - \mathrm{E}(X\,\mathrm{E}\,Y) - \mathrm{E}(Y\,\mathrm{E}\,X) + \mathrm{E}(\mathrm{E}\,X\,\mathrm{E}\,Y)\\ &= \mathrm{E}\,XY - \mathrm{E}\,X\,\mathrm{E}\,Y - \mathrm{E}\,Y\,\mathrm{E}\,X + \mathrm{E}\,X\,\mathrm{E}\,Y = \mathrm{E}\,XY - \mathrm{E}\,X\,\mathrm{E}\,Y\end{aligned}$$

ii) Obvious from above.

iii) From the Schwarz inequality (see bibliography).

iv) Follows straightforwardly by inserting into the definition.

v) Follows from the proof of the Schwarz inequality (see bibliography). □

Non-correlated random variables

Let us study the expectation of the product XY of two random variables X and Y.

Definition

Alternative definition: Two random variables X and Y are called **non-correlated** if

$$\mathrm{E}\, XY = \mathrm{E}\, X\, \mathrm{E}\, Y.$$

Lemma

If X and Y are independent then they are non-correlated.

Proof

Let X, Y be continuous variables. Independence means that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. Thus we have

$$\mathrm{E}\, XY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_X(x) f_Y(y)\, \mathrm{d}x\, \mathrm{d}y$$
$$= \left(\int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x \right) \left(\int_{-\infty}^{+\infty} y f_Y(y)\, \mathrm{d}y \right) = \mathrm{E}\, X\, \mathrm{E}\, Y.$$

□

Properties of the variance

It is now possible to obtain the following properties of the variance of sums of two random variables.

Theorem

i) *For X and Y with finite second moments:*

$$\operatorname{var}(X \pm Y) = \operatorname{var} X + \operatorname{var} Y \pm 2\operatorname{cov}(X, Y).$$

ii) *For non-correlated (independent) random variables it holds that*

$$\operatorname{var}(X \pm Y) = \operatorname{var} X + \operatorname{var} Y.$$

Properties of variance

Proof

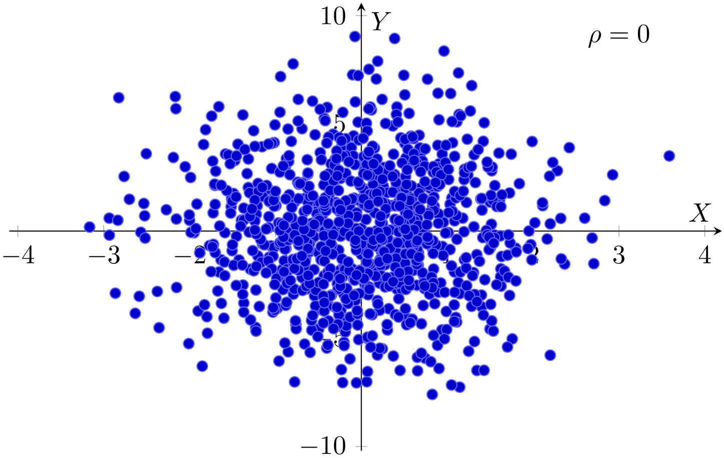
i) Given two random variables X and Y we have:

$$\begin{aligned}\operatorname{var}(X \pm Y) &= \mathrm{E}(X \pm Y)^2 - (\mathrm{E}(X \pm Y))^2 = \mathrm{E}(X^2 \pm 2XY + Y^2) - (\mathrm{E}\,X \pm \mathrm{E}\,Y)^2 \\ &= \mathrm{E}\,X^2 \pm 2\,\mathrm{E}\,XY + \mathrm{E}\,Y^2 - (\mathrm{E}\,X)^2 \mp 2\,\mathrm{E}\,X\,\mathrm{E}\,Y - (\mathrm{E}\,Y)^2 \\ &= \operatorname{var} X + \operatorname{var} Y \pm (2\,\mathrm{E}\,XY - 2\,\mathrm{E}\,X\,\mathrm{E}\,Y) = \operatorname{var} X + \operatorname{var} Y \pm 2\operatorname{cov}(X, Y).\end{aligned}$$

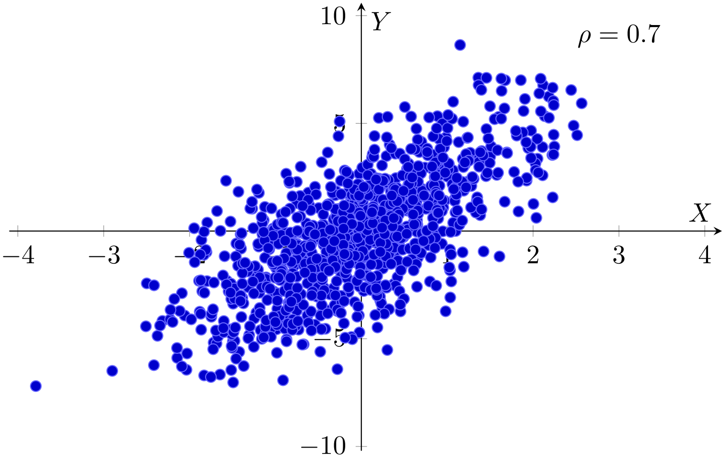
ii) For non-correlated (independent) random variables the covariance is zero.

□

Correlation – sample of 1000 values



Correlation – sample of 1000 values



Sums of random variables

An important case of a function of multiple random variables is their sum

$$Z = h(\boldsymbol{X}) = h(X_1, \dots, X_n) = X_1 + \cdots + X_n.$$

Consider for simplicity a sum of two random variables:

- If X and Y are **discrete** and **independent**, then for $Z = X + Y$ it holds that

$$\mathrm{P}(Z = z) = \sum_x \mathrm{P}(X = x) \cdot \mathrm{P}(Y = z - x) \quad \text{(discrete convolution)}.$$

- If X and Y are **continuous** and **independent**, then for $Z = X + Y$ it holds that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,\mathrm{d}x \quad \text{(convolution of } f_X \text{ and } f_Y\text{)}.$$

Sums of random variables – convolution (discrete case)

The expression for the sum of discrete independent X and Y is obtained easily:

$$\begin{aligned} \mathrm{P}(Z=z) &= \mathrm{P}(X+Y=z) \\ &= \sum_{\{(x_k,y_j):\, x_k+y_j=z\}} \mathrm{P}(X=x_k \cap Y=y_j) \\ &= \sum_{\text{all } x_k} \mathrm{P}(X=x_k)\ \mathrm{P}(Y=z-x_k). \end{aligned}$$

Sums of random variables - convolution (continuous case)

For continuous independent X and Y we have:

$$
\begin{aligned}
F_Z(z) = \mathrm{P}(X+Y\leq z) &= \iint\limits_{\{(x,y):\, x+y\leq z\}} f_{X,Y}(x,y)\,\mathrm{d}(x,y) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\overset{y=u-x}{=} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} f_{X,Y}(x,u-x)\,\mathrm{d}u\right)\mathrm{d}x \\
&= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,\mathrm{d}x\right)\mathrm{d}u \\
&= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f_X(x)f_Y(u-x)\,\mathrm{d}x\right)\mathrm{d}u.
\end{aligned}
$$

The density f_Z is any non-negative function, for which $F_Z(z)=\int_{-\infty}^{z} f_Z(u)\,\mathrm{d}u$.

The expression under the first integral $f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,\mathrm{d}x$ is thus the density of Z.

Sum of random variables – Normal distribution

Example – sum of two normal distributions

Suppose that X and Y are independent, both having the normal distribution $\mathrm{N}(\mu, 1)$. We want to obtain the distribution of $Z = X + Y$.

The densities of X and Y correspond to the normal distribution with variance $\sigma^2 = 1$:

$$f_X(x) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{(x-\mu)^2}{2\cdot 1}}, \qquad f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{(y-\mu)^2}{2\cdot 1}} \qquad x, y \in \mathbb{R}.$$

The density of the sum is obtained using convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x-\mu)^2}{2}}\,\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}\left((x-\mu)^2 + (z-x-\mu)^2\right)}\,\mathrm{d}x.$$

Sum of random variables – Normal distribution

Example – sum of two normal distributions, continuation

The expressions in the exponent can be rewritten as:

$$\begin{aligned}(x-\mu)^2+(z-x-\mu)^2 &= x^2-2\mu x+\mu^2+z^2+x^2+\mu^2-2zx-2\mu z+2\mu x\\ &= 2\left(x-\frac{z}{2}\right)^2+\frac{1}{2}(z-2\mu)^2.\end{aligned}$$

The expression under the integral can then be split into two multiplicative parts, with one of them not depending on x and the other one having an integral of 1:

$$\begin{aligned}f_Z(z) &= \int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\frac{2(x-z/2)^2}{2}}e^{-\frac{(z-2\mu)^2}{2\cdot 2}}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi 2}}e^{-\frac{(z-2\mu)^2}{2\cdot 2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1/2)}}e^{-\frac{(x-z/2)^2}{2\cdot(1/2)}}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi 2}}e^{-\frac{(z-2\mu)^2}{2\cdot 2}}.\end{aligned}$$

The sum $Z=X+Y$ has therefore the normal distribution $\mathrm{N}(2\mu,2)$. In general, it can be proven that the sum of n independent normals $\mathrm{N}(\mu,\sigma^2)$ has the distribution $\mathrm{N}(n\mu,n\sigma^2)$.

Sum of random variables – Poisson distribution

Example

Consider two independent random variables X and Y with the Poisson distribution with parameters λ_1 and λ_2, respectively. Find the distribution of the variable $Z = X + Y$.

$$\mathrm{P}(X = j) = \frac{\lambda_1^j}{j} e^{-\lambda_1} \qquad \mathrm{P}(Y = \ell) = \frac{\lambda_2^\ell}{\ell} e^{-\lambda_2}, \qquad j, \ell = 0, 1, \ldots$$

From what we have seen before we know that for $k = 0, 1, \ldots$:

$$\mathrm{P}(Z = k) = \sum_{\{(j,\ell)\in\mathbb{N}_0^2 :\, j+\ell=k\}} \mathrm{P}(X = j)\,\mathrm{P}(Y = \ell) = \sum_{i=0}^{k} \mathrm{P}(X = j)\,\mathrm{P}(Y = k - j)$$

$$= \sum_{j=0}^{k} \frac{\lambda_1^j}{j!} e^{-\lambda_1} \frac{\lambda_2^{k-j}}{(k-j)!} e^{-\lambda_2} = e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda_1^j \lambda_2^{k-j}$$

$$= \frac{(\lambda_1 + \lambda_2)^k}{k!} e^{-(\lambda_1+\lambda_2)}. \qquad \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

✓ An easier way is to use the moment generating function.

Sums of random variables – moment generating function

The moment generating function can be used to compute moments of random variables.

Taking a sum of independent random variables corresponds to taking a product of their generating functions:

For $Z = X + Y$ we have

$$M_Z(s) = \mathrm{E}(e^{sZ}) = \mathrm{E}(e^{s(X+Y)}) = \mathrm{E}(e^{sX}e^{sY})$$
$$= \mathrm{E}(e^{sX})\,\mathrm{E}(e^{sY}) = M_X(s)M_Y(s).$$

Generally for a vector of independent random variables $X_1, \dots, X_n$ it holds that:

$$Z = X_1 + \cdots + X_n \implies M_Z(s) = M_{X_1}(s) \cdots M_{X_n}(s).$$

Example

Let $X_1, \dots, X_n$ be independent **Bernoulli random variables** with parameter p.
Then $M_{X_i}(s) = (1-p)e^{0s} + pe^{1s} = 1 - p + pe^s, \quad i = 1, \dots, n.$

The random variable $Z = X_1 + \cdots + X_n$ is **binomial** with parameters n and p.
Its generating function is $M_Z(s) = \left(1 - p + pe^s\right)^n$.

Sum of random variables – moment generating function

Example

Let X and Y be independent **Poisson random variables** with parameters λ_1 and λ_2 respectively. Let $Z = X + Y$.

Then

$$M_Z(s) = M_X(s)M_Y(s) = e^{\lambda_1(e^s-1)}e^{\lambda_2(e^s-1)} = e^{(\lambda_1+\lambda_2)(e^s-1)}.$$

Z is again a Poisson random variable, this time with the parameter $\lambda_1 + \lambda_2$:

$$\mathrm{P}(Z = k) = \frac{(\lambda_1 + \lambda_2)^k}{k!} e^{-(\lambda_1+\lambda_2)}.$$

Compare with the difficulty of a direct computation of the convolution.

Summary

Joint distribution function of a random vector (X, Y):

$$F_{X,Y}(x, y) = \mathrm{P}(X \le x \cap Y \le y).$$

Discrete random variables X and Y	**Continuous random variables X and Y**

Joint probabilities of values / density:

$\mathrm{P}(X = x \cap Y = y)$	$f_{X,Y}(x, y)$

Marginal probabilities / density of X:

$\mathrm{P}(X = x) = \sum_y \mathrm{P}(X = x \cap Y = y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$
$\mathrm{P}(Y = y) = \sum_x \mathrm{P}(X = x \cap Y = y)$	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x$

Independence of X and Y:

$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y)$	$f_{X,Y}(x, y) = f_X(x) f_Y(y)$

Covariance of X and Y:

$$\operatorname{cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}\,X)(Y - \mathrm{E}\,Y)] = \mathrm{E}[XY] - \mathrm{E}\,X\,\mathrm{E}\,Y$$

X and Y are called non-correlated whenever $\operatorname{cov}(X, Y) = 0$.

If X and Y are independent, then they are also non-correlated.