Review - Random variables II: Moments

Basic characteristics of random variables

- The **expectation** (mean value) $\mathrm{E}\, X$, giving the mean (expected) value of X.
- The **variance** $\mathrm{var}\, X$, giving the dispersion (variability) of X around its mean $\mathrm{E}\, X$. The variance is defined as
$$\mathrm{var}\, X = \mathrm{E}(X - \mathrm{E}\, X)^2 = \mathrm{E}\, X^2 - (\mathrm{E}\, X)^2$$
and is always **non-negative**.

Discrete random variables:

For a discrete random variable with values $x_1, x_2, \ldots$:

- The expectation of X is computed as
$$\mathrm{E}\, X = \sum_k x_k \, \mathrm{P}(X = x_k) \quad \text{(if the sum exists);}$$
- The variance of X is computed as
$$\mathrm{var}\, X = \mathrm{E}\, X^2 - (\mathrm{E}\, X)^2 = \sum_k x_k^2 \, \mathrm{P}(X = x_k) - \left(\sum_k x_k \, \mathrm{P}(X = x_k) \right)^2 \quad \text{(if both sums exist);}$$
- The expectation of the random variable $Y = h(X)$ is computed as
$$\mathrm{E}\, Y = \mathrm{E}\, h(X) = \sum_k h(x_k) \, \mathrm{P}(X = x_k) \quad \text{(if the sum exists),}$$
or directly out of its distribution $\mathrm{E}\, Y = \sum_y y \, \mathrm{P}(Y = y)$.

Continuous random variables:

- The expectation of a continuous random variable X is computed as
$$\mathrm{E}\, X = \int_{-\infty}^{\infty} x f_X(x) dx \quad \text{(if the integral exists);}$$
- the expectation of $Y = h(X)$ is computed as
$$\mathrm{E}\, h(X) = \int_{-\infty}^{\infty} h(x) f(x) dx;$$
- in particular
$$\mathrm{E}\, X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx.$$

Useful properties

- The moment generating function of a random variable X is a function of $t \in \mathbb{R}$ and is defined as $M(t) = \mathrm{E}\, e^{tX}$. It holds that:
$$\mathrm{E}\, X = M'(0), \qquad \text{and} \qquad \mathrm{var}\, X = M''(0) - (M'(0))^2;$$
- when $a, b \in \mathbb{R}$ and X is a random variable, then
$$\mathrm{E}(a + bX) = a + b\, \mathrm{E}\, X, \qquad \text{and} \qquad \mathrm{var}(a + bX) = b^2 \, \mathrm{var}\, X;$$
- when $a, b \in \mathbb{R}$ and X and Y are random variables, then
$$\mathrm{E}(aX + bY) = a\, \mathrm{E}\, X + b\, \mathrm{E}\, Y.$$

Exercises 4 - Random variables II: Moments

1. Find the expected value and the variance of the random variable X denoting the number of Heads appearing after tossing two coins.
Remark: This random variable can be represented as a sum of the results of the tosses, where Heads is represented as 1, Tails as 0 and $\mathrm{P}(\text{Heads}) = p$.

2. We are rolling a six-sided die until a `6` occurs. Denote the moment when a `6` occurs for the first time as the X^{th} attempt. Find the expected number of rolls needed to get a six.

3. Suppose we are rolling a balanced n-sided die with values 1 to n. Let X be the number of points landed (*discrete uniform distribution*).

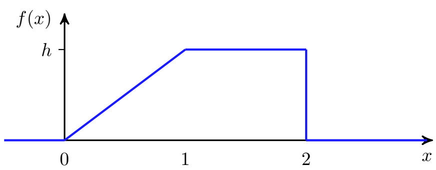
a) What is the expectation of X?

b) Is it possible to establish such a uniform distribution on values 1 to infinity?

4. Find the expected value and the variance of the random variable with probability density given by the following formula:

$$f_X(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } x \in (0,1] \\ 2-x & \text{for } x \in (1,2] \\ 0 & \text{for } x > 2. \end{cases}.$$

5. Let X be a continuous random devariable defined in Exercise 4 of Tutorial 3. Its density is given by the graph:



Find the expected value and the variance of X.

6. Suppose we observe a random variable X with $\mathrm{E}\,X = 1$ and $\operatorname{var} X = 2$. Find $\mathrm{E}(3X-4)$ and $\operatorname{var}(2X+1)$.

7. Let X be a random variable taking values $0, 1, 2, 3, 4, 5, 6, 7$ with the same probability. We define the random variable Y as

$$Y = \begin{cases} 1 & \text{for } X < \frac{1}{2} \\ 2 & \text{for } X \geq \frac{1}{2}. \end{cases}$$

Find the expected value and the variance of Y.

8. Suppose that a random variable X has the following properties:

$$\mathrm{E}\,X = 0, \qquad \mathrm{E}(X^2) = 1, \qquad \mathrm{E}(X^3) = 0, \qquad \mathrm{E}(X^4) = 2.$$

Random variables Y and Z are defined as

$$Y = 1 + X^2, \qquad \text{and} \qquad Z = 1 - X.$$

Find the expected values and variances of Y and Z.

Additional exercises - Random variables II: Moments

Discrete random variables

9. We have two 500 CZK banknotes, one 1000 CZK and one 2000 CZK in our pocket. A pickpocket reaches into the pocket and steals two banknotes at random. Let X be the random variable denoting the total value of our lost money.

a) Find the distribution of X.

b) Calculate the expected (mean) loss.

c) Calculate the variance of X.

d) Plot the distribution function of X.

10. Consider a lottery where a lottery ticket is a winning one with a probability p and a losing one with $1-p$. We devised a strategy of buying the tickets until we win.

a) Determine the distribution and expected number of losing tickets bought until we win.

b) Suppose that the winning award is $100,000$ CZK and one ticket costs 100 CZK. What is the minimum needed value of p for our strategy to pay off?

11. There are n gentlemen who left their hats at the cloakroom in a theater. After the play ends, the absent-minded cloakroom keeper gives each gentleman a hat selected at random. What is the expected number of gentlemen who got their own hats?

Transformations of random variables

12. Let X follow the Cauchy distribution with density

$$f_X(x) = c \cdot \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

a) Find c so that f_X truly is a density of a random variable.

b) Find the expectation $\operatorname{E} X$.

c) Find the density of $Y = 1/X$.

13. Suppose that the radius of a soap bubble in centimeters is a uniformly distributed random variable on the interval $[0,1]$. What is the distribution function, density and expectation of the volume of the bubble?

Quantile function

14. The time spent waiting for a tram forms a random variable X with the density

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}.$$

a) Find the quantile function F^{-1} of X and plot its graph.

b) Find the median of X and compare it to the expectation $\operatorname{E} X$.

c) Suppose U is a random variable uniformly distributed on the interval $[0,1]$. Find the distribution of $Y = F^{-1}(U)$.