Review - Random vectors I: independence, conditionals

Joint distribution:
If (X, Y) has a discrete distribution with values (x_i, y_j), the joint distribution is given by the probabilities $\mathrm{P}(X = x_i \cap Y = y_j)$ for all possible (x_i, y_j).

If (X, Y) is continuous with a density $f(x, y)$, then the joint distribution is given by the joint density $f_{XY}(x, y)$.

Marginal distribution:
If (X, Y) has a discrete distribution with values (x_i, y_j), then the marginal distribution of X is discrete and can be computed as

$$\mathrm{P}(X = x_i) = \sum_j \mathrm{P}(X = x_i \cap Y = y_j).$$

If (X, Y) is continuous with a density $f(x, y)$, then the marginal density of X is computed as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Similarly for the marginal distribution of Y.

Independence:
Two random variables X and Y are independent, if

$$\mathrm{P}(X \leq x \cap Y \leq y) = \mathrm{P}(X \leq x) \cdot \mathrm{P}(Y \leq y) \quad \forall\, (x, y) \in \mathbb{R}^2.$$

If (X, Y) has a continuous distribution then X and Y are independent if and only if $f_{XY}(x, y) = f_X(x) f_Y(y)$ for almost all $(x, y) \in \mathbb{R}^2$.

If (X, Y) has a discrete distribution then X and Y are independent if and only if $\mathrm{P}(X = x_i \cap Y = y_j) = \mathrm{P}(X = x_i)\, \mathrm{P}(Y = y_j)$ for all possible (x_i, y_j).

Conditional distribution:

a) If (X, Y) has a discrete distribution with values (x_i, y_j), then the conditional distribution of X given $Y = y$ is discrete and can be computed as

$$\mathrm{P}(X = x_i | Y = y_j) = \frac{\mathrm{P}(X = x_i \cap Y = y_j)}{\mathrm{P}(Y = y_j)}.$$

b) If (X, Y) is continuous with a density $f(x, y)$, then the conditional density of X given $Y = y$ is computed as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Conditional expectation:
If (X, Y) has a discrete distribution with values (x_i, y_j), then the conditional expectation of X given $Y = y$ is defined as

$$\mathrm{E}(X | Y = y_j) = \sum_{x_i} x_i\, \mathrm{P}(X = x_i | Y = y_j).$$

If (X, Y) has a continuous distribution, then the conditional expectation of X given $Y = y$ is defined as

$$\mathrm{E}(X | Y = y) = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\, \mathrm{d}x.$$

Exercises 6 - Random vectors I: independence, conditionals

1. Suppose that a random vector (X, Y) has the following joint probability distribution:

$\mathrm{P}(X = x \cap Y = y)$		Y: 0	1	2
X	1	0.4	0.15	0.05
	2	0.3	0.06	0.04

a) Find the marginal distributions of the random variables X and Y.

b) Are X and Y independent?

c) Find the conditional distribution of $X|Y = y$.

d) Find $\mathrm{E}(X|Y = y)$.

e) Find the conditional distribution of $X + Y$ given $Y = y$.

2. Consider two continuous random variables X and Y with joint density

$$f(x, y) = \begin{cases} \frac{1}{5}(4xy + 4x - 8y) & \text{for } 1 \le x \le 2 \text{ and } 0 \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find the conditional density $f_{Y|X}(y|x)$.

c) Find $\mathrm{E}(Y|X = x)$.

d) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

3. Consider two continuous random variables X and Y with the joint density

$$f(x, y) = \begin{cases} \frac{1}{2} y e^{(-x-y^2)/2} & \text{for } x > 0 \text{ and } y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

c) Find $\mathrm{E}(XY)$. *(See next Tutorial)*

4. Let X and Y be two continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{1}{3}(-4xy - 4x + 8y + 8) & \text{for } 1 \le x \le 2 \text{ and } 0 \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

c) Find $\mathrm{E}(XY)$. *(See next Tutorial)*

5. Let X and Y be continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{3}{8} x y^2 & \text{for } 0 \le x \le 1, \text{ and } -2 \le y \le 2\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the marginal density of the random variable X.

b) Find the conditional density $f_{Y|X}(y|x)$.

6. Suppose we roll two dice. Let X be the random variable defined as the sum of both rolled numbers and Y the random variable is defined as the difference between the number rolled on the first and the number rolled on the second dice. (i.e., $Y = 1^{st} - 2^{nd}$). Find $\mathrm{E}(X|Y = 2)$.

Additional exercises - Random vectors I: independence, conditionals

7. Two people are betting on the result of one roll of a six sided die. If a person is correct in their bet, they win 1. If the person is not correct, they lose 1. Suppose that person X bets on even numbers and person Y bets on large numbers $\{4, 5, 6\}$. Let X denote a random variable marking the winnings/losings of X after one game, same for Y.

a) Find the joint distribution of X and Y.

b) Find the marginal distributions for X and Y.

c) Are X and Y independent?

d) Find the conditional distribution of X given $Y = 1$ and given $Y = -1$.

e) Find the covariance and the coefficient of correlation between X and Y. *(See next Tutorial)*

8. Saint Martin is approaching, and in your restaurant you're preparing for a busy day. Let W the number of bottles of young wine drunk and G be the random variable describing the number of geese eaten, in an hour. Suppose that W and G are continuous with the joint density

$$f_{W,G}(x, y) = \begin{cases} \frac{1}{2}e^{-x-y/2} & \text{for } g > 0,\ w > 0, \\ 0 & \text{otherwise.} \end{cases}$$

a) What are the marginal distributions of W and G?

b) Are W and G independent?

c) After how much time are you expecting the first wine bottle to be emptied? And after how much time the first goose to be finished?

d) What is the covariance of W and G? *(See next Tutorial)*

e) Wht is the total expected time $E(W + G)$? *(See next Tutorial)*