Review - Random vectors II: correlation, convolution

Covariance and correlation:
The covariance $\operatorname{cov}(X, Y)$ of random variables X, Y is defined as

$$\operatorname{cov}(X,Y) = \mathrm{E}(X - \mathrm{E}\,X)(Y - \mathrm{E}\,Y) = \mathrm{E}(XY) - (\mathrm{E}\,X)(\mathrm{E}\,Y),$$

if $\mathrm{E}\,X^2 < \infty$ and $\mathrm{E}\,Y^2 < \infty$.
We compute

$$\mathrm{E}\,XY = \sum_{ij} x_i y_j \,\mathrm{P}(X = x_i \cap Y = y_j) \qquad \text{for discrete } X, Y,$$

and

$$\mathrm{E}\,XY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for continuous } X, Y.$$

The correlation coefficient ρ_{XY} is defined as

$$\rho_{XY} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}}$$

if $\operatorname{var} X, \operatorname{var} Y > 0$. Necessarily $-1 \le \rho_{XY} \le 1$. The correlation coefficient represents a measure of linear dependence between X and Y.

Note: If X and Y are independent, then $\operatorname{cov}(X, Y) = 0$. The reverse does not always hold.

Further properties: If $X_1, \dots, X_n$ are random variables, then (if exists):

- $\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{E}\,X_i,$
- $\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var} X_i + \sum\sum_{i \ne j} \operatorname{cov}(X_i, X_j).$

Sums of two random variables (convolution):

a) When X, Y are independent with a discrete distribution, then the random variable $Z = X + Y$ has a discrete distribution with probablities

$$\mathrm{P}(Z = n) = \sum_{k=-\infty}^{\infty} \mathrm{P}(X = k)\,\mathrm{P}(Y = n - k).$$

b) When X, Y are independent with a continuous distribution with densities f_X and f_Y, then the random variable $Z = X + Y$ has a continuous distribution with density

$$g(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,\mathrm{d}x.$$

Minimum and maximum:

a) If X and Y are independent, then

$$\mathrm{P}(\min(X,Y) > z) = \mathrm{P}(X > z \cap Y > z) = \mathrm{P}(X > z)\,\mathrm{P}(Y > z),$$
$$\mathrm{P}(\max(X,Y) \le z) = \mathrm{P}(X \le z \cap Y \le z) = \mathrm{P}(X \le z)\,\mathrm{P}(Y \le z).$$

b) For any X and Y it holds that:

$$\min(X,Y) + \max(X,Y) = X + Y.$$

Exercises 7 - Random vectors II: correlation, convolution

1. Variance of a sum

a) Show that for two non-correlated random variables X and Y it holds that
$$\operatorname{var}(X+Y) = \operatorname{var} X + \operatorname{var} Y.$$

b) Try to find a formula for the variance if X and Y are correlated.

c) Which of the previous formulas holds if X and Y are independent?

2. Let X_1 and X_2 be independent random variables with common expectation μ and variance σ^2.

a) Find the expected values and the variances of the random variables
$$Y_1 = X_1 + X_2, \qquad Y_2 = 2X_1, \qquad Y_3 = X_1 - X_2.$$

b) Compute the covariance $\operatorname{cov}(Y_i, Y_j)$ for $i, j = 1, 2, 3$, with $i \neq j$.

c) What do we know about the independence or the pairwise independence of the random variables Y_1, Y_2, Y_3?

3. Let X and Y be two independent random variables with geometric distribution with parameter p.

a) Find the distribution of $\min\{X, Y\}$.
Hint: *Compute* $\mathrm{P}(\min\{X, Y\} > k)$.

b) Find the expected value of $\max\{X, Y\}$.
Hint: $X + Y = \min\{X, Y\} + \max\{X, Y\}$.

c) Find the distribution of $\max\{X, Y\}$.
Hint: *Compute* $\mathrm{P}(\max\{X, Y\} \leq k)$.

d) Compute $\mathrm{P}(X < Y)$.
Hint: *Sum up the probabilities of favorable cases over all possible values of* X.

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables. Denote
$$Y = \min\{X_1, \ldots, X_n\}.$$

a) Suppose that all X_i have a discrete uniform distribution on $\{1, 2, \ldots, k\}$. Find the distribution of the random variable Y.

b) Suppose that all X_i have a geometric distribution with parameter p. Find the distribution of the random variable Y.

5. Let $X_1, X_2, \ldots, X_n$ be independent random variables. Denote
$$Y = \min\{X_1, \ldots, X_n\}.$$

Suppose that each X_i is exponentially distributed with parameter λ_i.

a) Find the distribution of the random variable Y. (Begin with $n = 2$)
Hint: *Compute* $\mathrm{P}(Y > k)$.

b) Find the expected value of $\max\{X_1, X_2\}$.
Hint: $X_1 + X_2 = \min\{X_1, X_2\} + \max\{X_1, X_2\}$.

6. Suppose that for two random variables X and Y it holds that:
$$\mathrm{E}\, X = 1, \qquad \mathrm{E}\, XY = 2.$$

Furthermore, suppose that the random variable Y is a linear function of X (i.e., $Y = aX + b$) and the variables X and Y are non-correlated. Find the variance of Y.

7. Consider two random variables X and Y and constants $a > 0$, b, $c > 0$, d. Prove that for the correlation coefficient it holds that:
$$\rho(aX + b, cY + d) = \rho(X, Y).$$

Additional exercises - Random vectors II: correlation, convolution

Random vectors continued

8. Suppose that X and Y are independent random variables with Poisson distribution with parameters λ_1 and λ_2 respectively. Find the distribution of $Z = X + Y$.

9. Suppose that X and Y are two independent random variables with exponential distribution with parameters λ_1 and λ_2 respectively. Find the distribution of $Z = \min(X, Y)$
Hint: *use the function* $S_Z(z) = \mathrm{P}(\min(X, Y) > z)$.

10. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (iid) continuous random variables with the same distribution function F and density f. Find the distribution function and density of $Z = \max(X_1, \ldots, X_n)$.

11. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (iid) continuous random variables with $\mathrm{E}\, X_1 = \mu$ and $\operatorname{var} X_1 = \sigma^2$. Denote $S_n = \sum_{i=1}^{n} X_i$. What is the expectation and variance of S_n?

12. Let X have an uniform distribution on the interval $[-1, 1]$. Denote $Y = X^2$. What are the covariance and the coefficient of correlation between X and Y? Are they independent?

13. * Suppose there are n gentlemen at a theatre, each putting their hat in the cloak room. On their way home, each of them gets one hat assigned at random. Consider random variables $X_1, \ldots, X_n$ with $X_i = 1$ if the i^{th} gentleman has his own hat and $X_i = 0$ if he doesn't.

a) Find the expectation and variance of X_i for a given i.

b) Are X_i and X_j independent for $i \neq j$? What is the covariance of X_i and X_j?

c) What is the mean and variance of the total number of correctly assigned hats $X = \sum_{i=1}^{n} X_i$?