Review - Point estimation

Theory of estimation:

- Random variables $X_1, \dots, X_n$ form a **random sample** if they are independent and all have the same distribution (i.i.d.).
  If we want to specify their distribution, we talk about a random sample from a distribution F (e.g., a random sample from $\mathrm{N}(\mu, \sigma^2)$, random sample from $\mathrm{Alt}(p)$, etc.)
- When $X_1, \dots, X_n$ is a random sample from a distribution depending on an **unknown** parameter $\theta \in \Theta$, then an **estimator** of θ can be any (measurable) function of $X_1, \dots, X_n$, if its function term does not depend on θ. The estimator $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ is therefore also a random variable.
- Numerical **estimates** of the parameters are then obtained by inserting actual measured data $x_1, \dots, x_n$ into the estimator.

Properties of estimators

- We want to use estimators with reasonable properties.
- We consider an estimator $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ of θ to be **unbiased** if

$$\mathrm{E}\, \hat{\theta}_n = \theta \qquad \text{for all } \theta \in \Theta.$$

- We consider an estimator $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ of θ to be **consistent** if

$$\hat{\theta}_n \xrightarrow{\mathrm{P}} \theta \qquad \text{for } n \to \infty$$

  for all $\theta \in \Theta$, meaning that $\mathrm{P}(|\hat{\theta}_n - \theta| > \varepsilon) \to 0$ for all $\varepsilon > 0$ and $\theta \in \Theta$.

Construction of pointwise estimates

- Maximum likelihood method (MLE): We find

$$\mathsf{argmax}_\theta \prod_{i=1}^{n} f_\theta(x_i) = \mathsf{argmax}_\theta \sum_{i=1}^{n} \log f_\theta(x_i),$$

  where $f_\theta(x_i)$ is the density of the random variables X_i for continuous distributions and $f_\theta(x_i) = \mathrm{P}(X_i = x_i)$ for discrete distributions.
- The method of moments: We estimate the moments $\mu_k = \mathrm{E}\, X^k$ using the empirical moments

$$\widehat{\mathrm{E}\, X^k} = m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

  If there is a link between the parameters and moments, say $\theta = g(\mathrm{E}\, X)$, we can estimate θ as $\hat{\theta} = g(\widehat{\mathrm{E}\, X})$. If we need to estimate more parameters, we need to use more empirical moments and solve a system of equations.

Exercises 9 - Point estimations

1. Show that the sample mean and sample variance are unbiased estimators of the expected value and variance, respectively.

2. Consider a realization of a random sample $X_1, \dots, X_{10}$:

$$4,\ 5.5,\ 7,\ 2.5,\ 6,\ 3,\ 8,\ 2,\ 3.5,\ 9.$$

Plot the empirical distribution function.

3. We want to estimate the number of carps in a pond. We proceed in the following way. First we catch 100 carps, tag them and release them back into the pond. After allowing time for the tagged carps to mix with the others, we catch a sample of 100 carps, 10 of which have tags.
Find an estimate for the total number of carps in the pond.

a) Find a rough estimate without any computation.

b) Use the maximal likelihood method. Let N be the number of carps in the pond. Find the probability that 10 of 100 caught fish have tags. Find such N that maximizes this probability (use a computer).

4. Suppose we observe a random sample of 10 pairs (X_i, Y_i). We have obtained the following:

$$\sum_{i=1}^{10} X_i = 10, \qquad \sum_{i=1}^{10} X_i^2 = 15, \qquad \sum_{i=1}^{10} Y_i = 4, \qquad \sum_{i=1}^{10} Y_i^2 = 7, \qquad \sum_{i=1}^{10} X_i Y_i = 6.$$

a) Find the sample means $\bar{X}_n$ and $\bar{Y}_n$ and the sample variances s_X^2 and s_Y^2.

b) Find the sample covariance

$$s_{X,Y} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n) = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i Y_i - n\bar{X}_n\bar{Y}_n\right).$$

c) Find the point estimate of the expected value of the random variable $Z = X + Y^2$.

5. Suppose that the length of a database server transaction is a random variable with exponential distribution with parameter λ. The lengths of transactions are independent.

a) Find the estimator for λ using the maximum likelihood method.

b) Find the estimator for λ using the method of moments.

From the server log we have the lengths of 10 transactions in milliseconds:

$$5.4,\ 15.6,\ 15.4,\ 9.3,\ 0.5,\ 14.4,\ 2.6,\ 0.7,\ 40.4,\ 21.9.$$

c) Find the actual value of the estimate of λ using both methods.

6. Find the maximum likelihood estimate for the expected value μ and the variance σ^2 of the normal distribution $\mathrm{N}(\mu, \sigma^2)$.

7. Let $X_1, \dots, X_n$ be i.i.d continuous random variables with density

$$f(x, a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the estimators of the parameters a and b using the method of moments.

Additional exercises - Point estimation

Pointwise parameter estimation

8. Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution with parameter $\lambda > 0$.

a) Find the estimator $\widehat{\lambda}_n$ of λ using the method of moments.

b) Is $\widehat{\lambda}_n$ unbiased?

c) Is $\widehat{\lambda}_n$ consistent?

d) Find the estimator $\widehat{\lambda}_n$ of λ using the maximum likelihood method.

e) Consider the estimator $\hat{\lambda}_n^{(2)} = (X_1 + X_2)/2$. Is this estimator unbiased and consistent?

9. Let $X_1, \dots, X_n$ be a random sample from the exponential distribution $\mathrm{Exp}(\lambda)$.

a) Find the estimator $\widehat{\lambda}_n$ of λ using the maximum likelihood method.

b) Is $\widehat{\lambda}_n$ consistent?

10. Suppose we run a random number generator, generating numbers from a uniform distribution on the interval $[0, b]$, with b unknown. Let $X_1, \dots, X_n$ be a random sample from this distribution.

a) Find an estimator of $\hat{b}_n$ of b. There are at least two reasonable estimators utilizing all data.

b) Is $\hat{b}_n$ unbiased?

11. Let $X_1, \dots, X_n$ be a random sample from the normal distribution $\mathrm{N}(\mu, \sigma^2)$ with density

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}} \qquad \text{for } x \in \mathbb{R},$$

with both parameters μ and σ^2 unknown. Find the maximum likelihood estimators of both μ and σ^2.