Elementary Introduction to Graph Theory

(01EIG 2025/2026)

Lecture 1

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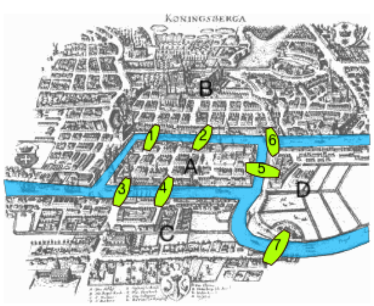
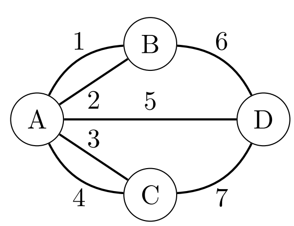
A Swiss mathematician and seven Prussian bridges

In 1736 the mayor of the Prussian city of Königsberg asked the mathematician Leonard Euler whether it was possible to take a walk around the city crossing each of the seven bridges exactly once.

Euler's intuition was to describe the problem by means of a diagram consisting of a set of points and lines instead of regions and bridges.

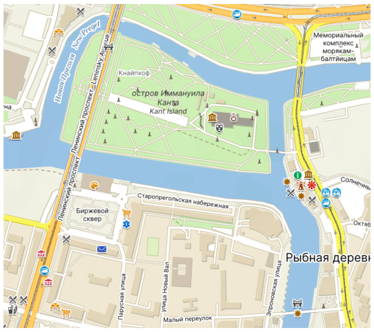


(a) Königsberg in Euler's time.

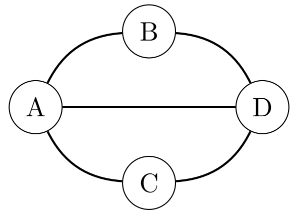


(b) Diagram of Königsberg.

History passed on Königsberg, now known as Kaliningrad (or Královec), and some of the bridges fell down.



(a) Kaliningrad nowadays.



(b) Diagram of Kaliningrad.

Was the walk possible before? Is it now?

1 Basic definitions

Let V be a set and $r \in \mathbb{N}_0$. By $\binom{V}{r}$ we denote the set of all r-element subsets of V. Note that $\#\binom{V}{r} = \binom{\#V}{r}$, where $\#A$ is the cardinality of A.

Example 1 If $V = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, then

$$\binom{V}{0} = \{\emptyset\}, \qquad \binom{V}{1} = \{\{\mathtt{a}\}, \{\mathtt{b}\}, \{\mathtt{c}\}\},$$

$$\binom{V}{2} = \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{c}\}, \{\mathtt{b}, \mathtt{c}\}\}, \qquad \binom{V}{3} = \{V\}$$

A *graph* is a pair $G = (V, E)$, where V is a set of *vertices* and E is a multiset of *edges* of the form $e = \{u, v\}$, with $u, v \in V$. The numbers $\#V$ and $\#E$ are respectively the *order* and the *size* of G. A graph G is *finite* if both $\#V$ and $\#E$ are finite.

The usual way to illustrate a graph is to draw a dot for each $v \in V$ (or a circle with the label v), and to join $u, v \in V$ by a line if $e = \{u, v\} \in E$.

Example 2 The order of the graph $G = (\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{b}, \mathtt{c}\}\})$ is 3, while its size is 2 (see left of Figure 3).

The order of the graph $G' = (\{\mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{x}, \mathtt{y}\}, \{\{\mathtt{u}, \mathtt{w}\}, \{\mathtt{v}, \mathtt{x}\}, \{\mathtt{v}, \mathtt{x}\}, \{\mathtt{y}, \mathtt{y}\}\})$ is 5, while its size is 4 (see right of Figure 3).

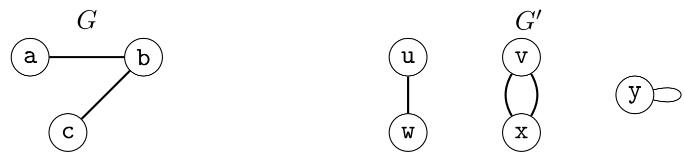


Figure 3: Two graphs.

Graphs can be represented graphically, but there is not a unique way to do so: relative position of points, shape/length of lines, etc. We are only interested in the incidence relation between vertices and edges.

Two vertices connected by an edge are said to be *adjacent*. They are both *incident* with the edge. Two distinct adjacent vertices are *neighbours*. Given a vertex $v \in V$ we define its neighbourhood in G

$$\mathrm{N}_G\left(v\right) = \{u \mid u \text{ neighbour of } v\} = \{u \mid \{u, v\} \in E\}$$

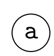
An edge of the form $\{u, u\}$ is called a *loop*. Two edges connecting the same vertices are called *multiedges* (or *parallel edges*) A graph $G = (V, E)$ is *simple* if it does not contain any loop or multiedge, i.e., if $E \subseteq \binom{V}{2}$.

The graph $(\emptyset, \emptyset)$ is called the *null graph*.

We will mostly consider finite non-empty simple graphs.

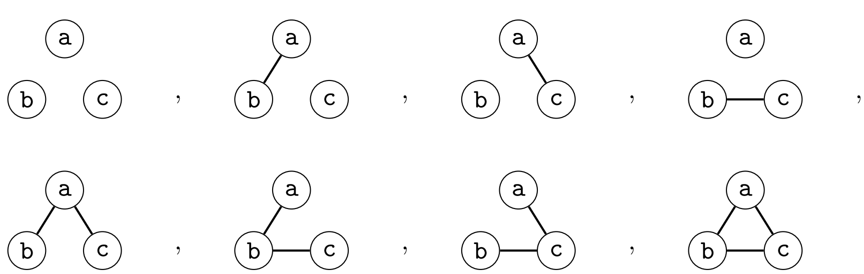
Example 3 How many simple graphs are there on n vertices? $2^{\binom{n}{2}}$.

- The only graph of 0 vertices is the null graph: $2^{\binom{0}{2}} = 2^0 = 1$.
- If $V = \{\mathtt{a}\}$, there is only one possible graph: $2^{\binom{1}{2}} = 2^0 = 1$



- If $V = \{\mathtt{a}, \mathtt{b}\}$, there are two possible graphs: $2^{\binom{2}{2}} = 2^1 = 2$

- If $V = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, there are eight possible graphs: $2^{\binom{3}{2}} = 2^3 = 8$



2 Some important graphs

Any graph of the form $(\{v\}, \emptyset)$, i.e., with only one edge and no edges, is called a *trivial graph*. Every other graph is called *non-trivial*.

A simple graph of the form $K_n = \left(V, E = \binom{V}{2}\right)$, with $\#V = n$, is called a *complete graph* (see Figure 4).

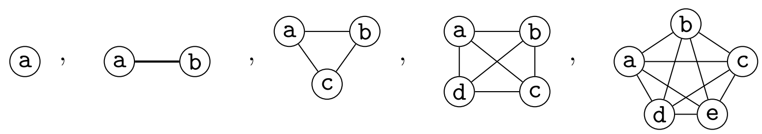


Figure 4: Complete graphs K_1, K_2, K_3, K_4 and K_5.

A graph $G = (V, E)$ is *bipartite* if the set of vertices is a disjoint union $V = X \sqcup Y$ and every edge $e \in E$ is of the form $e = \{x, y\}$ with $x \in X$ and $y \in Y$. When , $\#X = m$ and $\#Y = n$, and we are connecting every edge of X with every edge of Y, we call such a graph *complete bipartite* and denote it $K_{n,m}$ (see Figure 5). A graph of the form $K_{1,k}$ is called *k-star*.

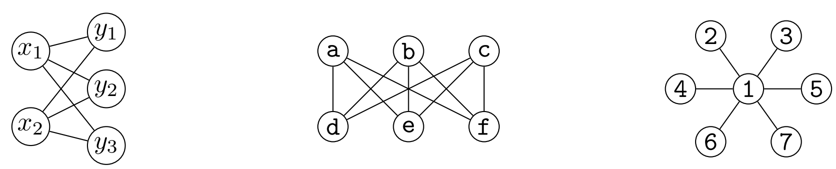


Figure 5: Three bipartite graphs.

3 Graph operations

If we have two graphs $G = (V, E)$ and $H = (U, F)$ we can combine them in several ways to obtain new graphs.

Their *union* is defined as $G \cup H = (V \cup U, E \cup F)$. Note thati f $V \cap U = \emptyset$ we just obtain a "juxtaposition" of the two graphs.

Their *intersection* is defined as $G \cap H = (V \cap U, E \cap F)$.

Example 4 Let us consider $G = (\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}, \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{d}\}, \{\mathtt{b}, \mathtt{c}\}, \{\mathtt{c}, \mathtt{d}\}\})$ and $H = (\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{e}\}, \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{c}\}, \{\mathtt{b}, \mathtt{e}\}, \{\mathtt{c}, \mathtt{e}\}\})$ (see Figure 6). Then

$$G \cup H = (\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}, \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{c}\}, \{\mathtt{a}, \mathtt{d}\}, \{\mathtt{b}, \mathtt{c}\}, \{\mathtt{b}, \mathtt{e}\}, \{\mathtt{c}, \mathtt{d}\}, \{\mathtt{c}, \mathtt{e}\}\})$$

and

$$G \cap H = (\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\{\mathtt{a}, \mathtt{b}\}\}) .$$

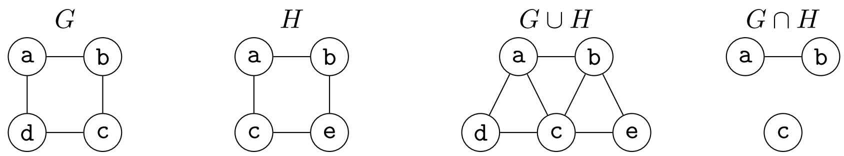


Figure 6: Union and intersection of two graphs.

The *cartesian product* $G \,\square\, H$ of the two graphs is defined as the graph having set of vertices $V \times U$ and edges defined by

$$\{(a,u),(b,v)\} \text{ edge of } G \,\square\, H \Leftrightarrow ((\{a,b\} \in E \wedge u = v) \;\vee\; (a = b \wedge \{u,v\} \in F))\,.$$

Example 5 Let $G = (\{\mathtt{a},\mathtt{b},\mathtt{c}\},\{\{\mathtt{a},\mathtt{b}\},\{\mathtt{a},\mathtt{c}\},\{\mathtt{b},\mathtt{c}\}\})$ and $H = (\{\mathtt{1},\mathtt{2}\},\{\{\mathtt{1},\mathtt{2}\}\})$. The cartesian product $G \,\square\, H$ is represented in Figure 7.

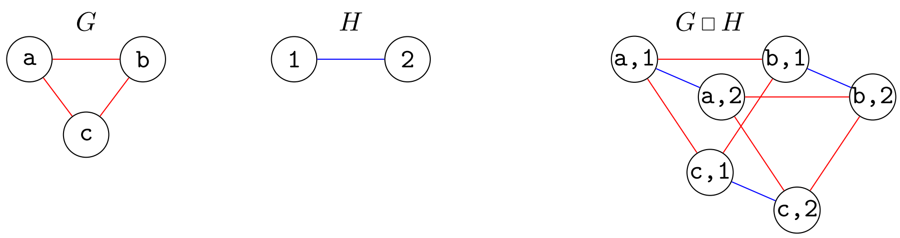


Figure 7: Cartesian product of two graphs.

If two graphs $G = (V, E_1)$ and $H = (V, E_2)$ are defined over the same set of vertices, we can define their *symmetric difference* $G \bigtriangleup H$ as

$$G \bigtriangleup H = (V, E_1 \bigtriangleup E_2), \qquad \text{where} \quad E_1 \bigtriangleup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)\,.$$

Example 6 Let $G = (V, E_1)$ and $H = (V, E_2)$,where $V = \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}$, $E_1 = \{\{\mathtt{a},\mathtt{b}\},\{\mathtt{a},\mathtt{d}\},\{\mathtt{b},\mathtt{c}\},\{\mathtt{c},\mathtt{d}\}\}$, and $E_2 = \{\{\mathtt{a},\mathtt{b}\},\{\mathtt{b},\mathtt{d}\},\{\mathtt{c},\mathtt{d}\}\}$. Then $G \bigtriangleup H = (V,\{\{\mathtt{a},\mathtt{d}\},\{\mathtt{b},\mathtt{d}\},\{\mathtt{b},\mathtt{c}\}\})$ (see Figure 8).

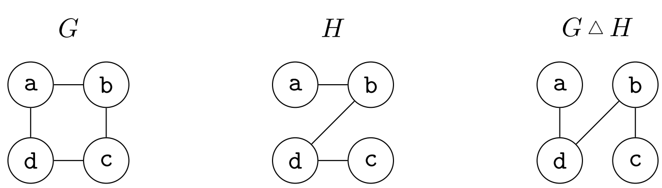


Figure 8: Symmetric difference of two graphs.

4 Degree of vertices

Let $G = (V, E)$ be a simple graph. The *degree* of a vertex $v \in V$ in G is

$$d_G(v) = \#\mathrm{N}_G(v) = \#\{u \in V \mid \{v, u\} \in E\},$$

i.e., the number of edges in G incident with v. When G is clear from the context, we just write $d(v)$. The *minimum degree* of G is $\delta(G) = \min\{d_G(v) \mid v \in V\}$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V\}$. The *average degree* of G is $d(G) = \frac{1}{\#V} \sum_{v \in V} d_G(v)$. If $d_G(v) = 0$, we say that v is an *isolated vertex* in G.

Example 7 Let G be the graph in Figure 9. We have

$$\delta(G) = d(\mathtt{f}) = 0, \qquad d(\mathtt{a}) = d(\mathtt{b}) = d(\mathtt{c}) = d(\mathtt{d}) = 1, \qquad \Delta(G) = d(\mathtt{e}) = 4,$$

$$d(G) = \frac{1+1+1+1+4+0}{6} = \frac{4}{3}.$$

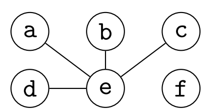


Figure 9: A graph with minimal degree 0 and maximal degree 4.

The definition can be extended to non simple graphs by counting parallel edges separately. In this case each loop counts twice.

Lemma 1 (Handshaking Lemma) *Let $G = (V, E)$ be a graph. Then,*

$$\sum_{v \in V} d_G(v) = 2\#E.$$

Proof. We sum up all vertices multiplied by their degree. We count every edge exactly twice, one for each of its ends. ∎

Corollary 1 *Let $G = (V, E)$ be a graph. Then, $\#\{v \mid d_G(v)$ is odd$\}$ is even.*

Proof. For each $v \in V$ we have

$$d_G(v) \equiv \begin{cases} 1 \pmod 2 & \text{if } d_G(v) \text{ is odd,} \\ 0 \pmod 2 & \text{if } d_G(v) \text{ is even.} \end{cases}.$$

Thus, $\sum_{v \in V} d_G(v) \equiv \#$ vertices of odd degree $\pmod 2$.

By the Handshaking Lemma 1, $\sum_{v \in V} d_G(v) \equiv 0 \pmod 2$. So, the number of vertices with odd degree is also even. ∎

Example 8 Let G be the graph in Example 7. The sum of the degrees is 8 and we have 4 vertices of odd degree: `a,b,c,d`.

A graph $G = (V, E)$ is called *k-regular* if $d_G(v) = k$ for every $v \in V$.

Example 9 K_n is $(n-1)$-regular for every $n \in \mathbb{N}_0$. A 3-regular graph is called a *cubic graph* (see Figure 10).

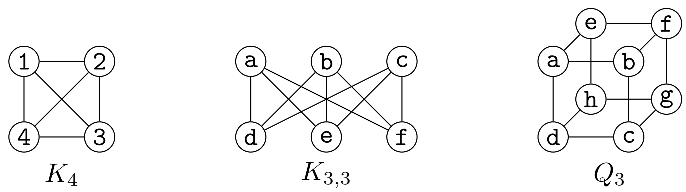


Figure 10: Three cubic graphs.

A graph in which each vertex has even degree is called an *even graph*.

Example 10 The graph K_2 (Figure 4) is even, while the graph $K_{1,6}$ (Figure 5) is not.

5 Graphic sequences

A sequence $(d_i)_{i=1}^n$ is called *graphic* if there exists a simple graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ such that for every $1 \leq i \leq n$ we have $d_G(v_i) = d_i$. In this case we say that G *realises* the sequence $(d_i)_{i=1}^n$.

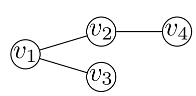
We usually order the sequence s.t. $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$.

Clearly not every sequence of non-negative integers is graphic. The Handshaking Lemma 1 gives us a necessary condition: $\sum_{i=1}^n d_i$ must be even. Also, in this case we must have $d_1 \leq n-1$.

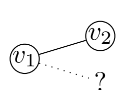
Example 11

- The sequence $(2, 2, 1, 1)$ is graphic.

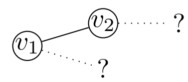
  A graph realising it is $G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}\})$.



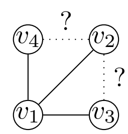
- The sequence $(2, 1)$ is not graphic. Indeed $2 + 1 = 3$ is not even.



- The sequence $(2,2)$ is not graphic. Indeed $d_1 = 2 > 2 - 1 = 1$.



- The sequence $(3,3,1,1)$ is not graphic (why?).



Theorem 1 (Havel (1955), Hakimi (1961)) *A non-increasing non-negative sequence* $(d_i)_{i=1}^n$ *is graphic if and only if*

$$(d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n)$$

is graphic.

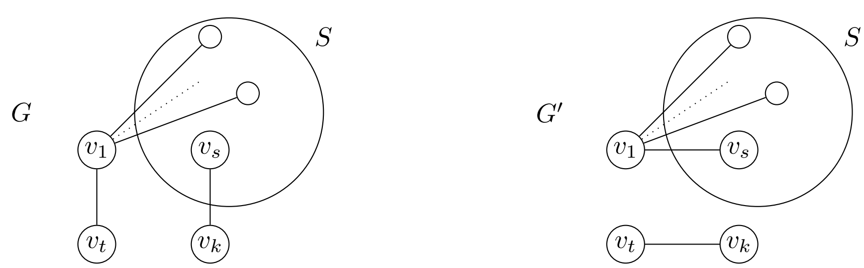
To prove Havel-Hakimi Theorem we need the following result.

Lemma 2 *Let* $(d_i)_{i=1}^n$ *be a graphic sequence with* $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. *Then, there is a simple graph* $G = (V, E)$ *realising* $(d_i)_{i=1}^n$ *such that* $V = \{v_1, \ldots, v_n\}$, $d_G(v_i) = d_i$ *for every* i, *and* $\mathrm{N}_G(v_1) = \{v_2, \ldots, v_{d_1+1}\}$.

Proof. Let $S = \{v_2, \ldots, v_{d_1+1}\}$. From all graphs realising $(d_i)_{i=1}^n$ let us select G s.t. $r = |\mathrm{N}_G(v_1) \cap S|$ is maximum.

If $r = d_1$, then G is the graph we are looking for and we are done.

If, by contradiction, $r < d_1$, then there exist s, t with $1 \leq s \leq d_1 + 1 < t \leq n$ such that $\{v_1, v_t\} \in E$ and $\{v_1, v_s\} \notin E$. Since $d_G(v_s) \geq d_G(v_t)$ and $\{v_1, v_t\} \in E$, there exists v_k such that $\{v_s, v_k\} \in E$ and $\{v_t, v_k\} \notin E$. Let $G' = (G \setminus \{\{v_1, v_t\}, \{v_s, v_k\}\}) + \{\{v_1, v_s\}, \{v_t, v_k\}\}$.



We have $|\mathrm{N}_{G'}(v_1) \cap S| = r + 1 > r$, which contradict the maximality of r. ■

Proof of Havel-Hakimi Theorem.

(⇒) If $(d_i)_i$ is graphical, by Lemma 2 there exists a graph G realising it with $\mathrm{N}_G(v_1) = \{v_2, \ldots, v_{d_1+1}\}$. Thus $G \setminus \{v_1\}$ has degree sequence $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$.

(⇐) If $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphical, there exists a graph $G' = (V', E')$ with $\#V' = n - 1$, $V' = \{v_2, \ldots, v_n\}$ such that

$$d_{G'}(v_i) = \begin{cases} d_i - 1 & \text{for } 2 \leq i \leq d_{d_1+1} \\ d_i & \text{for } d_{d_1+2} \leq i \leq n \end{cases} .$$

Thus the graph $G = \left(V' \cup \{v_1\},\ E' + \bigcup_{i=2}^{d_1+1} \{v_1, v_i\} \right)$ realises $(d_i)_{i=1}^n$.

■

Havel-Hakimi Theorem gives us a recursive algorithm to check whether a sequence is graphic.

Algorithm 1: `GraphicSequence(`$d_1, d_2, \ldots, d_n$`)`

Input: non-increasing sequence $(d_1, d_2, \ldots, d_n)$
Output: `TRUE` if $(d_i)_{i=1}^n$ is graphic; `FALSE` if not

if *$(d_1 > n - 1)$ or $(d_n < 0)$* **then**
  return `FALSE`
else if $d_1 = 0$ **then**
  return `TRUE`
else
  Let $(a_1, a_2, \ldots, a_{n-1})$ non-increasing permutation of
   $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$
  `GraphicSequence`$(a_1, a_2, \ldots, a_{n-1})$

Example 12

- $(2, 1)$ is not a graphic sequence, since $2 > 1$.
- $(\mathbf{3}, \underline{2, 1, 1}) \to (\mathbf{1}, \underline{0}, 0) \to (-1, 0) \sim (0, -1)$ is not graphic since $-1 < 0$.
- $(\mathbf{2}, \underline{2, 1}, 1) \to (1, 0, 1) \sim (\mathbf{1}, \underline{1}, 0) \to (0, 0)$ is graphic (see Figure 11).

We can also use the algorithm backwards to realise a graphic sequence: we start with as many vertices are there are 0s in the last sequence and then, at each step, add a vertex with edges according to its degree (see Figure 11).

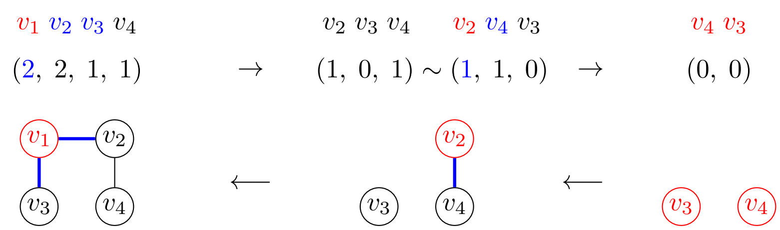


Figure 11: Realising a graph from a graphic sequence.

Exercise. Determine which of the following sequences are graphic, and in the positive case find a graph realising them: $(7,6,5,4,3,3,2)$, $(3,3,2,2,1,1)$.

We also have the following non recursive result.

Theorem 2 (Erdős, Gallai (1960)) *A sequence $(d_i)_{i=1}^n$ with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ is graphic if and only if*

$$\sum_{i=1}^{n} d_i \text{ is even} \qquad \text{and} \qquad \sum_{i=1}^{k} d_i < k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \quad \forall\, 1 \leq k \leq n.$$