Elementary Introduction to Graph Theory

(01EIG 2025/2026)

Lecture 9

Francesco Dolce
dolcefra@fit.cvut.cz

November 28, 2025

updated: November 28, 2025

PDF available at the address: dolcefra.pages.fit/ens/2526/EIG-lecture-09.pdf

Solution of Exercise in previous Lecture.
We apply the Hungarian algorithm, starting from the matching $M_0 = \emptyset$.

1. Consider the empty matching $M_0 = \emptyset$.

   (a) Let us apply Step 1 by choosing the vertex a and labelling it S.

   (b) Let us apply Step 2a) by choosing the edge $\{$a, b$\}$ and labelling the vertex b as L.

   We find a new M_0-incrementing path $P_1 = ($a, b$)$ (see Figure 1).

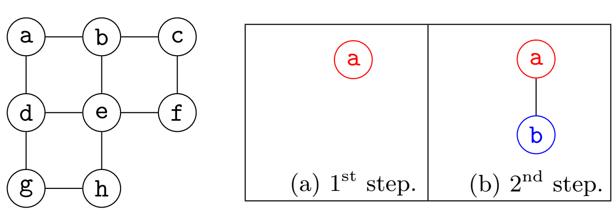

Figure 1: First run of the Hungarian algorithm on G.

We can use this path to find a new matching

$$M_1 = M_0 \triangle P_1 = \{\{\mathtt{a}, \mathtt{b}\}\}.$$

2. Consider the matching $M_1 = \{\{\mathtt{a}, \mathtt{b}\}\}$.

   (a) Let us apply Step 1 by choosing the vertex c and labelling it S.

   (b) Let us apply Step 2b) by choosing the edge $\{\mathtt{c}, \mathtt{b}\}$, with b matched via the edge $\{\mathtt{b}, \mathtt{a}\} \in M_1$; we label the vertex b as L and the vertex a as S.

   (c) Let us apply Step 2a) starting from c, by choosing the edge $\{\mathtt{c}, \mathtt{f}\}$; we label the vertex f as L.

   We find a new M_1-incrementing path $P_2 = (\mathtt{c}, \mathtt{f})$ (see Figure 2).

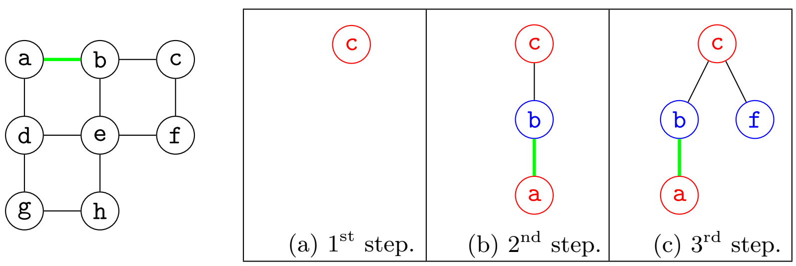

Figure 2: Second run of the Hungarian algorithm on G.

We can use this path to find a new matching

$$M_2 = M_1 \triangle P_2 = \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{c}, \mathtt{f}\}\}.$$

3. Consider the matching $M_2 = \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{c}, \mathtt{f}\}\}$.

   (a) Let us apply Step 1 by choosing the vertex d and labelling it S.

   (b) Let us apply Step 2b) by choosing the edge $\{\mathtt{d}, \mathtt{a}\}$, with a matched via the edge $\{\mathtt{a}, \mathtt{b}\} \in M_2$; we label the vertex a as L and the vertex b as S.

   (c) Let us apply Step 2a) starting from b, by choosing the edge $\{\mathtt{b}, \mathtt{e}\}$; we label the vertex e as L.

   We find a new M_2-incrementing path $P_3 = (\mathtt{d}, \mathtt{a}, \mathtt{b}, \mathtt{e})$ (see Figure 3).

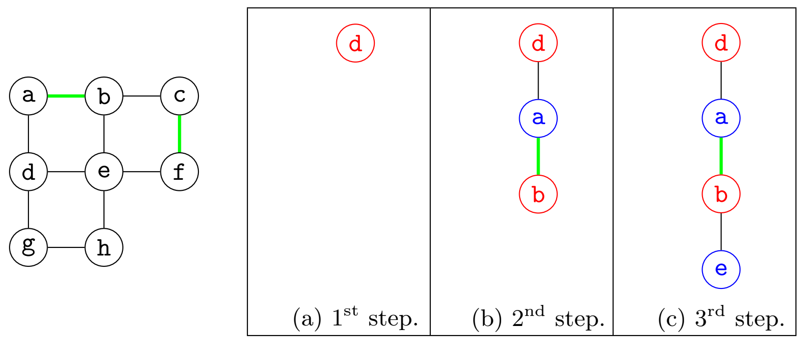

Figure 3: Third run of the Hungarian algorithm on G.

We can use this path to find a new matching

$$M_3 = M_2 \triangle P_3 = \{\{\mathtt{a}, \mathtt{d}\}, \{\mathtt{b}, \mathtt{e}\}, \{\mathtt{c}, \mathtt{f}\}\}.$$

4. Consider the matching $M_3 = \{\{\mathtt{a}, \mathtt{d}\}, \{\mathtt{b}, \mathtt{e}\}, \{\mathtt{c}, \mathtt{f}\}\}$.
   (a) Let us apply Step 1 by choosing the vertex `g` and labelling it `S`.
   (b) Let us apply Step 2b) by choosing the edge $\{\mathtt{g}, \mathtt{d}\}$, with `d` matched via the edge $\{\mathtt{d}, \mathtt{a}\} \in M_3$; we label the vertex `d` as `L` and the vertex `a` as `S`.
   (c) Let us apply Step 2b) by choosing the edge $\{\mathtt{a}, \mathtt{b}\}$, with `b` matched via the edge $\{\mathtt{b}, \mathtt{e}\} \in M_3$; we label the vertex `b` as `L` and the vertex `e` as `S`.
   (d) Let us apply Step 2a) starting from `e`, by choosing the edge $\{\mathtt{e}, \mathtt{h}\}$; we label the vertex `h` as `L`.

   We find a new M_3-incrementing path $P_4 = (\mathtt{g}, \mathtt{d}, \mathtt{a}, \mathtt{b}, \mathtt{e}, \mathtt{h})$ (see Figure 4).

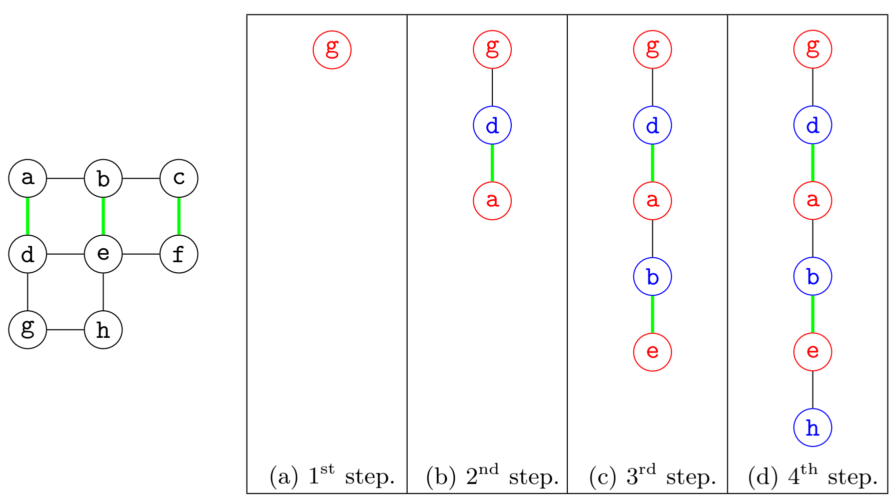

Figure 4: Fourth run of the Hungarian algorithm on G.

We can use this path to find a new matching

$$M_4 = M_3 \triangle P_4 = \{\{\mathtt{a},\mathtt{b}\},\{\mathtt{c},\mathtt{f}\},\{\mathtt{d},\mathtt{g}\},\{\mathtt{e},\mathtt{h}\}\}.$$

The matching M_4 is maximal (and perfect) in G (see Figure 5).

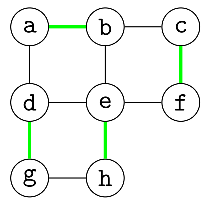

Figure 5: A maximal matching in G.

Solution of Exercise in previous Lecture. The problem consists in finding a 7-edge-colouring of the graph K_8. A possible solution is given in Figure 6. The edge-colourings $\varphi_1, \varphi_2, \ldots, \varphi_8$ are such that for every i we have

$$\varphi_1^{-1}(E) = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\},$$
$$\varphi_2^{-1}(E) = \{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\},$$
$$\varphi_3^{-1}(E) = \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\},$$
$$\varphi_4^{-1}(E) = \{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\},$$
$$\varphi_5^{-1}(E) = \{\{1,6\},\{2,5\},\{3,8\},\{4,7\}\},$$
$$\varphi_6^{-1}(E) = \{\{1,7\},\{2,8\},\{3,5\},\{4,6\}\},$$
$$\varphi_7^{-1}(E) = \{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}.$$

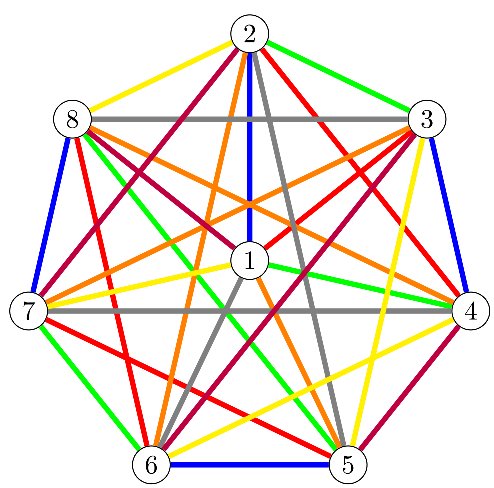

Figure 6: A 7-edge colourings of the graphs K_8.

1 Vertex colouring

Let $G = (V, E)$ be a graph. A *vertex colouring* by k colours, or *vertex k-colouring*, of G is a mapping $\varphi : V \to \hat{k}$, with k a positive integer. Similarly to edge-colouring, a vertex-colouring φ is called *proper* if for every two distinct vertices $u, v \in V$ we have

$$\{u, v\} \in E \quad \Rightarrow \quad \varphi(u) \neq \varphi(v),$$

i.e., vertices connected by an edge have different colours.

A graph admitting a vertex k-colouring is called *k-colourable*.

Example 1 Let $G = (V, E)$ be the graph in Figure 7, and $\varphi_1, \varphi_2 : V \to \{1, 2, 3\}$ and $\varphi_3 : V \to \{1, 2, 3, 4, 5\}$ be the three vertex-colourings of G defined by

$$\varphi_1(\mathtt{a}) = \varphi_1(\mathtt{d}) = 1, \quad \varphi_1(\mathtt{b}) = 2 \quad \text{and} \quad \varphi_1(\mathtt{c}) = \varphi_1(\mathtt{e}) = 3;$$

$$\varphi_2(\mathtt{a}) = 1, \quad \varphi_2(\mathtt{b}) = \varphi_2(\mathtt{e}) = 2 \quad \text{and} \quad \varphi_2(\mathtt{c}) = \varphi_2(\mathtt{d}) = 3;$$

$$\varphi_3(\mathtt{a}) = 1, \quad \varphi_3(\mathtt{b}) = 2, \quad \varphi_3(\mathtt{c}) = 3, \quad \varphi_3(\mathtt{d}) = 4, \quad \text{and} \quad \varphi_3(\mathtt{e}) = 5.$$

The first and last vertex-colourings are proper, while the second one is not, since, e.g., $\varphi_2(\mathtt{b}) = \varphi_2(\mathtt{e}) = 2$.

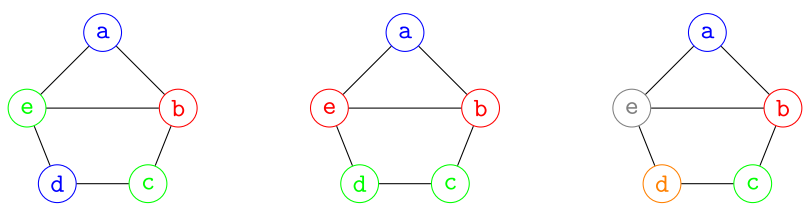

Figure 7: A graph with three different vertex-colouring.

The *chromatic number* of G, denoted by $\chi(G)$, is the smallest $k \in \mathbb{N}$ such that G has a proper vertex colouring using k colours (also called proper vertex k-colouring). A graph having chromatic number k is called *k-chromatic*.

Example 2 The graph G in Figure 8, known as the *Hajós graph*, has chromatic number $\chi(G) = 4$. Indeed, it is possible to show that no proper vertex 3-colouring exists, while a proper vertex 4-colouring is given by

$$\varphi(\mathtt{a}) = \varphi(\mathtt{d}) = 1, \quad \varphi(\mathtt{b}) = \varphi(\mathtt{e}) = 2, \quad \varphi(\mathtt{c}) = \varphi(\mathtt{f}) = 3 \quad \text{and} \quad \varphi(\mathtt{g}) = 4.$$

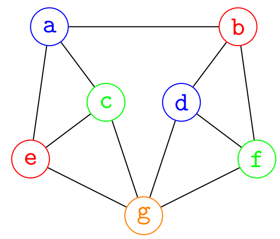

Figure 8: The Hajós graph.

Example 3 (Examination Scheduling) Let us assume that we have m students who are to take n exams during an examining period. Lecturers aim to schedule the exams in such a way that there are are no conflicts, i.e., they seek a conflict-free schedule with the minimum number of time slots.

We can construct a graph $G = (V, E)$ with

- set of vertices V given by the exams, i.e., $V = \{z_1, z_2, \ldots, z_n\}$,
- set of edges E given by connecting exams z_i and z_j if there is a student who is supposed to take both z_i and z_j.

The chromatic number of such a graph gives the minimal number of time slots needed for a conflict-free scheduling of exams.

An example is given in Figure 9, where the set of vertices (exams) is partitioned by a 4-colouring φ as

$$\varphi^{-1}(1) = \{z_1, z_3\}, \quad \varphi^{-1}(2) = \{z_2\} \quad \text{and} \quad \varphi^{-1}(3) = \{z_4, z_5\}.$$

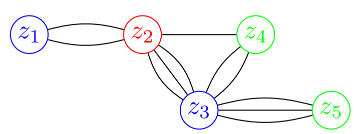

Figure 9: A graph representing the scheduling of exams.

Example 4 (Chemical Storage) A company manufactures n chemicals c_1, c_2, …, c_n. Certain pairs of these chemicals are incompatible - they would cause an explosion if brought into contact with each other.

The company wishes to partition its warehouse into compartments so that incompatible chemicals are stored separately. What is the least number of compartments into which the warehouse should be partitioned?

The solution is very similar to the exam scheduling in Example 3.

We construct a graph $G = (V, E)$, with vertices $V = \{c_1, c_2, \ldots, c_n\}$ by joining c_i and c_j with an edge in E if and only if the chemicals c_i and c_j are incompatible.

It is easy to see that the required least number of compartments is equal to the chromatic number of G.

An example is given in Figure 10 where the set of vertices is given by the chemicals **Na** (Sodium), **Cl** (Chlorine), **K** (Potassium), **F** (Fluorine), **Li** (Lithium) and $\mathbf{H_2O}$ (Water) and a possible vertex-colouring is given by

$$\varphi(\mathbf{Na}) = \varphi(\mathbf{K}) = \varphi(\mathbf{Li}) = 1, \quad \varphi(\mathbf{Cl}) = \varphi(\mathbf{H_2O}) = 2 \quad \text{and} \quad \varphi(\mathbf{F}) = 3.$$

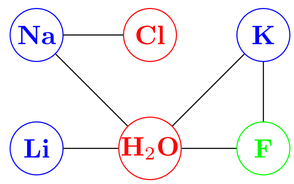

Figure 10: A graph representing chemicals not to be stored together.

Clearly only loopless graphs admit proper vertex-colourings. Moreover, a graph is k-colourable if and only if its underlying simple graph if k-colourable. Thus, in the rest of this section we can focus only on simple graphs.

Unfortunately, no good algorithm is known for determining the chromatic number of a generic graph. Only for a few kind of graphs we know their chromatic numbers.

The following Proposition is immediate, since a trivial graph has only one vertex and no edges.

Proposition 1 $\chi((V, \emptyset)) = 1$.

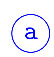

Figure 11: A colouring of the trivial graph.

Similarly, we have the following trivial result for the regular graph K_n, where each pair of vertices is connected by an edge.

Proposition 2 $\chi(K_n) = n$.

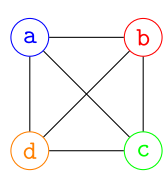

Figure 12: A colouring of the graph K_4.

This is actually a characterisation, since for every graph $G = (V, E)$ of order n that is not the complete graph, there exists at least a pair of vertices $u, v \in V$, such that $\{u, v\} \notin E$. So we can colour u and v with the same colour.

Bipartite graphs are exactly the ones who are 2-colourable. That is, we have the following result

Proposition 3 *$\chi(G) = 2$ if and only if G is a bipartite graph.*

Thus, it is quite easy to check whether a graph G on n vertices has

- $\chi(G) = 1 \quad \Leftrightarrow \quad$ is there an edge in G?
- $\chi(G) = n \quad \Leftrightarrow \quad$ is $G = K_n$?
- $\chi(G) = 2 \quad \Leftrightarrow \quad$ is G bipartite? $\quad \Leftrightarrow \quad$ Does G contain an odd cycle? (There is a linear-time algorithm to check the existence of an odd cycle).

On the other hand, determining whether $\chi(G) = 3$ for a given graph G is a $\mathcal{NP}$-complete problem (the class of most "difficult" decision problems).

There is a quite simple *greedy algorithm* which can be used to find a proper colouring of G. It is a linear-time algorithm, but it does not – in general – give the minimum number of colours possible.

The algorithm processes the vertices in the given order, assigning colours successively: each vertex is given the colour (from the set $\{1, 2, 3, \ldots\}$), with the smallest number that is not already used by one of its neighbours.

The resulting colouring depends on the ordering of vertices.

Example 5 Consider the following colouring of the same graph G, where the vertices are relabeled and thus reordered (equivalently, we consider two isomorphic graphs).

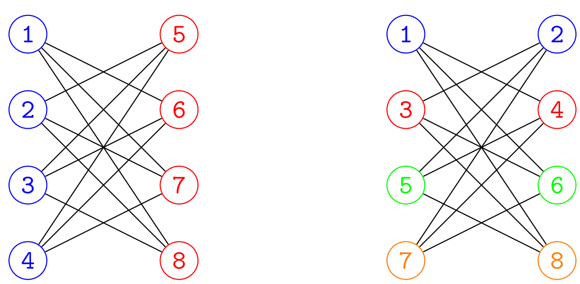

Figure 13: A graph with two different vertex-colouring.

With the first ordering, we obtain the 2-colouring $\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$, while with the second we obtain the 4-colouring $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$.

Even though there always exists an ordering of vertices that produces a proper colouring, such ordering is (in general) hard to find.

A commonly used strategy is to place high-degree vertices earlier than low-degree ones.

The greedy algorithm at least gives us an upper bound of the chromatic number of G: in every step the considered vertex has at most $\Delta(G)$ neighbours. Thus at most $\Delta(G)$ colours are "blocked", there is always a proper vertex colouring of G using $\Delta(G)+1$ colours.

Theorem 1 *Let $G=(V,E)$ be a graph. Then $\chi(G) \leq \Delta(G)+1$.*

On the one hand, this upper bound is tight, since $\chi(K_n)=n$ and $\Delta(K_n)=n-1$. On the other hand, there are graphs for which the difference $\Delta(G)+1-\chi(G)$ can be arbitrary large.

Example 6 Let $K_{1,n}$ be the n-star. Since $K_{1,n}$ is bipartite, its chromatic number is $\chi(K_{1,n})=2$, but $\Delta(K_{1,n})=n$.

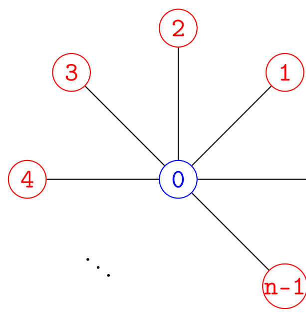

Figure 14: The n-star $K_{1,n}$.

Exercise. The graph in Figure 15 is called *Chvátal graph*. It is a 4-regular graph with 12 vertices. Show that the graph is 4-chromatic.

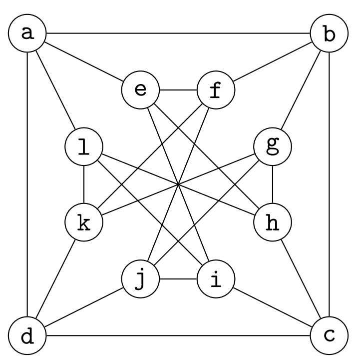

Figure 15: The Chvátal graph.

2 Clique and stability number

There are other known bound on the chromatic number of a graph. To state them, let us give a couple of more definitions.

A *clique* in a graph $G = (V, E)$ is a set $S \subseteq V$ such that $\binom{S}{2} \subseteq E$, i.e., every pair of distinct vertices in S is adjacent (connected by an edge).

Clearly every vertex $v \in V$ forms a clique $\{v\}$. Similarly, for ever edge $\{u, v\} \in E$ we have that $\{u, v\}$ is a clique.

The *clique number* of G, denoted $\omega(G)$, is the size of the largest clique in G.

Example 7 Consider the complete graph K_4. The sets $\{\mathtt{a}\}$, $\{\mathtt{b}, \mathtt{c}, \mathtt{d}\}$ (see left of Figure 16) are cliques in K_4 (see rleft of Figure 16). Since the graph is regular, also the set $V = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ is a clique (see right of Figure 16), so the clique number of the graph is $\omega(K_4) = 4$.

Figure 16: Three cliques of the complete graph K_4.

In general, the clique number of K_n is $\omega(K_n) = n$ for every $n \in \mathbb{N}$.

Example 8 Let G be the Hajós graph seen in Example 2. The sets $\{\mathtt{b}\}$, $\{\mathtt{d}, \mathtt{f}\}$, $\{\mathtt{a}, \mathtt{c}, \mathtt{e}\}$ are cliques in G (see Figure 17). One can check that we do not have any clique of size 4, so the clique number of the graph is $\omega(G) = 3$.

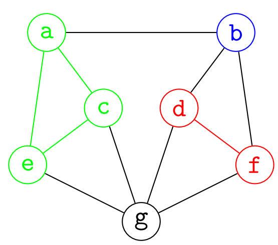

Figure 17: Three cliques of the Hajós graph.

A *stable set* (also called *independent set*) in $G = (V, E)$ is a set $S \subseteq V$ such that $\binom{S}{2} \cap E = \emptyset$, that is, no two vertices in S are adjacent.

The *stability number* (or *independence number*) of G, denoted $\alpha(G)$, is the number of vertices in a largest stable set in G.

Example 9 Consider the complete graph K_4. The set $\{\mathtt{a}\}$ is a stable set of K_4. Since every vertex is adjacent to every other vertex, the largest stable set has size 1, i.e., $\alpha(K_4) = 1$.

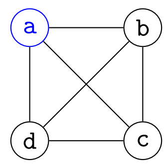

Figure 18: A stable set in the complete graph K_4.

In general, the stability number of K_n is $\omega(K_n) = 1$ for every $n \in \mathbb{N}$.

Example 10 Let G be the Hajós graph seen in Example 2. The sets $\{\mathtt{b}\}$, $\{\mathtt{a}, \mathtt{d}\}$, $\{\mathtt{c}, \mathtt{f}\}$ (Figure 19) are stable sets in G. One can check that we do not have any stable set of size bigger or equal to 3, so the stability number of the graph is $\alpha(G) = 2$.

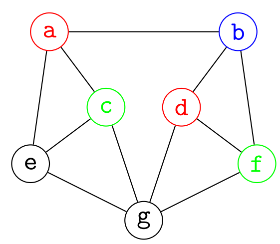

Figure 19: Three stable sets of the Hajós graph.

There is a quite obvious relation between proper colourings and independent sets.

Let φ be a proper k-vertex colouring of G. Then, the set $\varphi^{-1}(i)$, for $i \in \hat{k}$ (i.e., the set of vertices coloured by the colour i) is an independent set in G. Thus for every $i \in \hat{k}$ we have $\#\varphi^{-1}(i) \leq \alpha(G)$, which gives the following result.

Theorem 2 *Let $G = (V, E)$ be a graph. Then, $\#V(G) \leq \alpha(G) \cdot \chi(G)$.*

Proof. Consider a minimal proper colouring φ of G. Then,

$$\#V = \sum_{i=1}^{\chi(G)} \#\varphi^{-1}(i) \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \alpha(G) \cdot \chi(G).$$

∎

The set $\varphi^{-1}(i)$ is usually called a *colour class*.

The connection between colourings and cliques is even simpler. Since any clique induces a subgraph of G which is complete, any proper colouring of G

need to use k colours to colour vertices of a clique of size k. The following bound easily follows.

Theorem 3 *Let G be a graph. Then, $\chi(G) \geq \omega(G)$.*

3 Brooks Theorem and critical graphs

We have seen before that the upper bound $\chi(G) \leq \Delta(G) + 1$ is tight, i.e., there are graphs for which $\chi(G) = \Delta(G) + 1$.

The example used to demonstrate the tightness was K_n.

There is another class of graphs for which the upper bound is reached: the class of odd cycles C_{2n+1}, where C_n is the cycle graph on n vertices, i.e., n vertices connected in a cycle.

Obviously, for every $n \in \mathbb{N}$, we have $\chi(C_{2n+1}) = 3 = \Delta(C_{2n+1}) + 1$.

Example 11 Consider the cycle C_7 (see Figure 20). Each vertex has degree 2, so $\Delta(C_7) = 2$. On the other hand we need three colours to properly colour all vertices in the graph, e.g.,

$$\varphi(1) = \varphi(3) = \varphi(5) = 1, \quad \varphi(2) = \varphi(4) = \varphi(6) = 2 \quad \text{and} \quad \varphi(7) = 3.$$

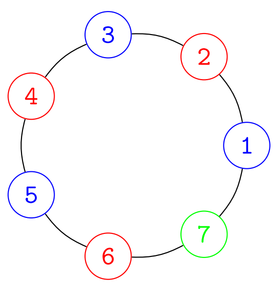

Figure 20: A colouring of the graph C_7.

The following theorem states that complete graphs and odd cycles are the only graphs which require $\Delta + 1$ colours.

Theorem 4 (Brooks (1941)) *Let $G = (V, E)$ be a connected graph which is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

We will not inspect the proof of Brooks' Theorem as it is quite long and (at least a part of it) quite technical. Instead, we will only mention a class of graphs heavily used in the proof, since they are quite interesting and useful on their own.

A graph G is called *k-critical* if its chromatic number is $\chi(G) = k$, and for any proper subgraph H of G we have $\chi(H) \leq k - 1$.

The only 1-critical graph is K_1. Similarly, the only 2-critical graph is K_2.

Since a proper subgraph of a cycle is a union of paths, we have that C_{2n+1} is 3-critical for any $n \in \mathbb{N}$. Odd cycles are actually the only possible simple graphs which are 3-critical.

Proposition 4 *The only* 3*-critical simple graphs are the odd cycles* C_{2n+1}.

No equivalent characterisation of k-critical graphs is known for $k = 4$.

There are methods to construct a k–critical graph for a given $k \in \mathbb{N}$.

Example 12 Consider the two graphs in Figure 21 (the one on the right is known as the *Grötzsch graph*). One can check that both graphs are 4-critical.

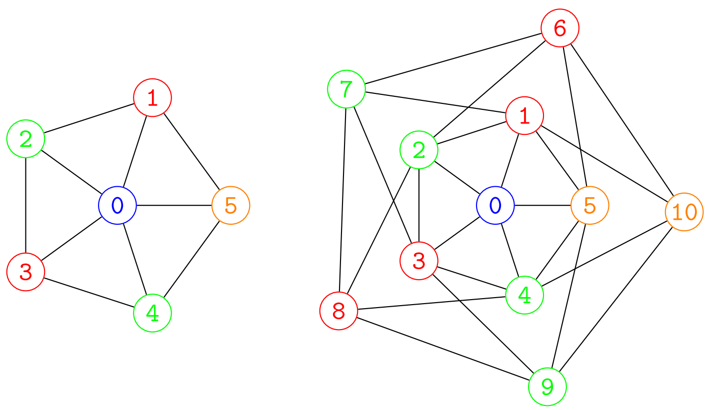

Figure 21: Two 4-critical graphs

Exercise. Show that the Hajós graph (defined in Example 2) is 4-critical.

Proposition 5 *Let* G *be a* k*-critical graph. Then* G *is connected and* $\delta(G) \geq k - 1$.