Elementary Introduction to Graph Theory

(01EIG 2025/2026)

Lecture 10

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Solution of Exercise in previous Lecture.

The Chvátal graph does not have any vertex 3-colouring since it is 4-regular. A possible vertex 4-colouring is shown in Figure 1 and is given by

$$\varphi^{-1}(1) = \{\texttt{a}, \texttt{c}, \texttt{f}, \texttt{g}\},\ \varphi^{-1}(2) = \{\texttt{b}, \texttt{d}, \texttt{e}, \texttt{h}\},\ \varphi^{-1}(3) = \{\texttt{i}, \texttt{k}\},\ \varphi^{-1}(4) = \{\texttt{j}, \texttt{l}\}.$$

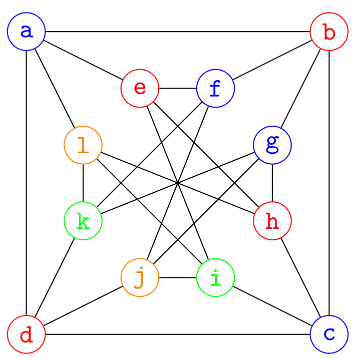


Figure 1: The Chvátal graph with a vertex 4-colouring.

Solution of Exercise in previous Lecture. The possible subgraphs of the Hajós graph with one vertex removed are shown in Figure 2. One can check that each of them admits a 3-colouring. All other proper subgraphs of the Hajós graph are subgraphs of one of these seven graphs, so they also admit a 3-colouring.

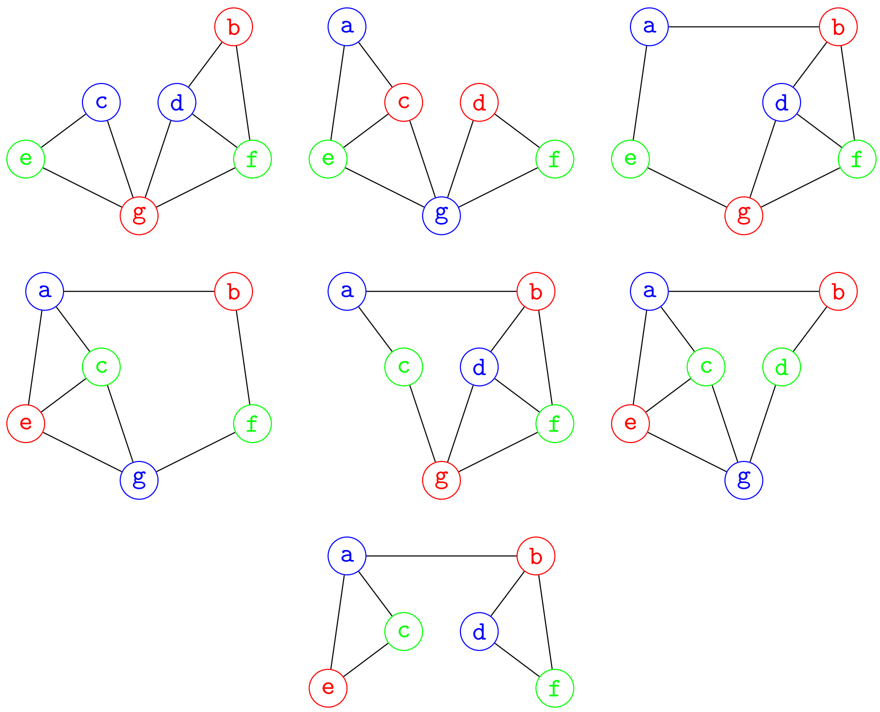


Figure 2: Maximal subgraphs of the Hajós graph.

1 Planar graphs

When we draw a graph on a piece of paper, we naturally try to do this as transparently as possible. One obvious way to limit the mess created by all the lines is to avoid intersections, that is, we try to draw the graph in such a way that two edges do not meet in a point other than a common vertex.

A graph is said to be *planar* if it can be drawn (formally, it is identical or isomorphic to a graph drawn) in the plane without crossing of the edges. That means that two edges meet only at their common vertex.

More formally, a graph $G = (V, E)$ is called *planar* if there are two bijective mappings $\varphi : V \to \mathbb{R}^2$ and $\psi : E \to \mathcal{C}$, where $\mathcal{C}$ is the set of simple, continuous curves in $\mathbb{R}^2$, such that

(P1) for every $e = \{u, v\} \in E$, we have that $\varphi(u)$ and $\varphi(v)$ are endpoints of $\psi(e)$;

(P2) for every distinct $e, f \in E$, we have that $\psi(e) \cap \psi(f) = \varphi(e \cap f)$.

Example 1 The graph K_4 is planar. A representation where vertices and edges satisfy the two conditions (P1) and (P2) is shown in Figure 3

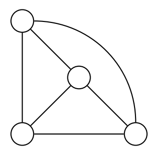


Figure 3: The graph K_4 is planar.

To study planar graphs we need some reminder of topology.

Recall that a *curve* in the real plane is just a continuous image of a segment, while a *closed curve* is a continuous image of a circle. A curve in $\mathbb{R}^2$ is called *simple* if it does not cross itself.

Example 2 The curve on the left and the closed curve in the centre of Figure 4 are simple, while the closed curve on the right of the same figure is not.

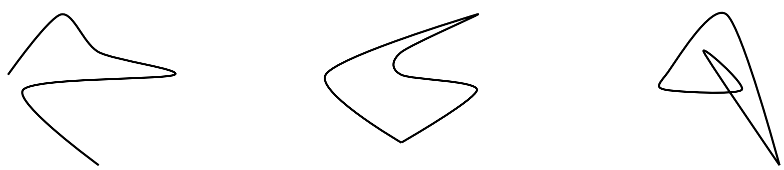


Figure 4: A curve (left) and two closed curves (centre and right) in $\mathbb{R}^2$.

Open sets in the real plane can be seen as a generalisation of open intervals in the real line. A simple definition of an *open set* in $\mathbb{R}^2$ is: a set $M \subseteq \mathbb{R}^2$ such that every point in it is the centre of an open circle (circle without its boundary) contained in M.

A set $M \subseteq \mathbb{R}^2$ is *arcwise-connected* if for every $x, y \in M$ there exists a continuous function $f : [0, 1] \to M$ such that $f(0) = x$ and $f(1) = y$.

Example 3 Let us consider the set M defined as union of two disjoint open circles, i.e., we are considering only the points "inside" the circles and not on the borders. The set M is open but not arcwise-connected, since there is no arc in M connecting a point inside the first circle to a point inside the second one (see, e.g., Figure 5, where no curve inside M can have as endpoints x and z).

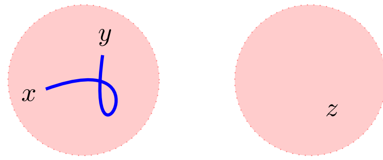
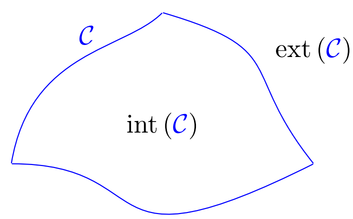


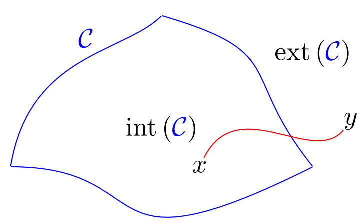
Figure 5: An open but not arcwise-connected set.

An essential result of topology that we are using when talking about planar graphs is the following.

Theorem 1 (Jordan Curve Theorem) *Any simple closed continuous curve $\mathcal{C}$ in $\mathbb{R}^2$ partitions the rest of $\mathbb{R}^2$ into two disjoint arcwise-connected open sets, denoted int(C), or* interior *of $\mathcal{C}$, and ext$(\mathcal{C})$, or* exterior *of $\mathcal{C}$.*



Corollary 1 *Let $\mathcal{C} \in \mathbb{R}^2$ be a a simple, closed, continuous curve (called* Jordan curve*). Every continuous curve connecting a point $x \in$ int$(\mathcal{C})$ with a point $y \in$ ext$(\mathcal{C})$ intersect $\mathcal{C}$ somewhere.*

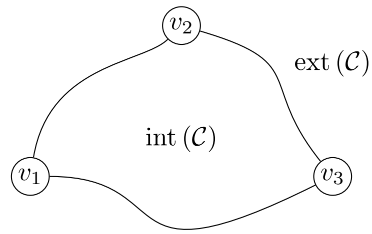


Using the previous corollary it is possible to prove that the regular graph K_5 is not planar.

Proposition 1 *The graph K_5 is not planar.*

Proof. Let $K_5 = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$.

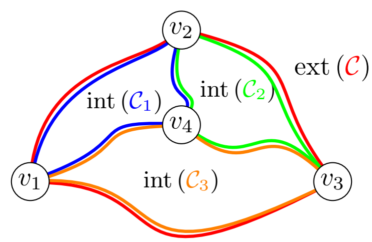
Assume, by contradiction, that K_5 is planar. Then, there is a planar drawing of K_5. First, let us consider the cycle (v_1, v_2, v_3). It corresponds to a Jordan curve $\mathcal{C}$ in our plane drawing of K_5.



Now, there are two possibilities for the placement of v_4: either $v_4 \in \text{int}(\mathcal{C})$, or $v_4 \in \text{ext}(\mathcal{C})$.

If we place v_4 into $\text{int}(\mathcal{C})$ and connect it by curves representing the edges $\{\{v_4, v_1\}, \{v_4, v_2\}, \{v_4, v_3\}\}$ we have the following closed curves:

- $\mathcal{C}$ given by the cycle (v_1, v_2, v_3),
- $\mathcal{C}_1$ given by the cycle (v_1, v_2, v_4),
- $\mathcal{C}_2$ given by the cycle (v_2, v_3, v_4),
- $\mathcal{C}_3$ given by the cycle (v_3, v_1, v_4).



We now have 4 possible placements of v_5: in $\text{int}(\mathcal{C}_1)$, in $\text{int}(\mathcal{C}_2)$, in $\text{int}(\mathcal{C}_3)$, or in $\text{ext}(\mathcal{C})$. None of them is possible in a planar drawing. For example, if $v_5 \in \text{int}(\mathcal{C}_1)$, since $v_3 \in \text{ext}(\mathcal{C}_1)$, the curve representing the edge $\{v_1, v_5\}$ would intersect (by the Jordan Curve Theorem 1) the curve $\mathcal{C}_1$, and thus the drawing would not be planar.

All other options – $v_5 \in \text{int}(\mathcal{C}_2)$, or in $\text{int}(\mathcal{C}_3)$, or in $\text{ext}(\mathcal{C})$; as well as v_4 in $\text{ext}(\mathcal{C})$ – lead to similar contradictions. ∎

When discussing about plane drawnings of a planar graph, we have to specify whether or not we are considering (only) planar drawings, since planar graphs (can) admit non-planar drawings as well.

Example 4 The graph on the left of Figure 6 is planar, but it has also non-planar drawings.

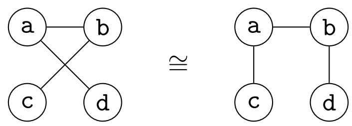


Figure 6: A planar graph.

A planar drawing of a (planar) graph is also called a *plane graph*.

The following classic results of Euler – here stated in its simplest form, for the plane $\mathbb{R}^2$ – marks one of the origins of connections between graph theory and topology. The theorem relates the number of vertices, edges and faces, i.e., regions of $\mathbb{R}^2 \setminus G$, of a planar graph G. There is a general version of the formula which asserts a similar equality for graphs embedded in other surfaces (e.g., on a sphere).

Given a plane graph G, a *face* of G is an arcwise-connected region of $\mathbb{R}^2$ which borders are given by edges in G. Every planar graph has one unbounded face, called the *outer face*.

Theorem 2 (Euler's formula) *Let $G = (V, E)$ be a connected planar graph. Then*

$$\#V - \#E + \Psi(G) = 2,$$

where $\Psi(G)$ is the number of faces of a planar drawing of G.

As a consequence of the previous theorem we have that every planar drawing of a given planar graph has the same number of faces.

Example 5 A planar drawing of K_4 with four faces, denoted $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 (this last being the outer face) is given in Figure 7. We have

$$\#V - \#E + \Psi(G) = 4 - 6 + 4 = 2.$$

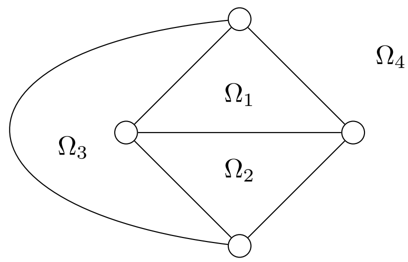


Figure 7: A planar graph with four faces.

We can use Euler's formula to derive two simple necessary (but not sufficient) conditions for the planarity of a graph G.

Proposition 2 *Let $G = (V, E)$ be a planar graph with $\#V = n \geq 3$.*

1) $\#E \leq 3n - 6$.

2) If G is bipartite, then $\#E \leq 2n - 4$.

Example 6 The complete graph K_4 has 4 vertices and $6 \leq 3 \cdot 4 - 6$ edges, according to point 1) in Proposition 2. The bipartite graph in Figure 8 has 6 vertices and $8 \leq 2 \cdot 6 - 4$ edges, according to point 2) of the same proposition.

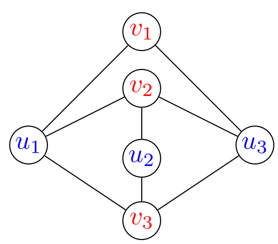


Figure 8: A bipartite graph.

Corollary 2 *Every simple planar graph has a vertex of degree at most five.*

Proof. Let $G = (V, E)$ be a simple planar graph, with $\#V = n$ and $\#E = m$. Then, using Proposition 2 and the results seen in previous lectures we have,

$$\delta(G) \cdot n \ \leq \ \sum_{v_i \in V} d_G(v) \ = \ 2m \ \leq \ 2 \cdot (3n - 6) \ = \ 6n - 12.$$

Thus $\delta(G) \leq 6 - \frac{12}{n} < 6$, which implies $\delta(G) \leq 5$. ∎

We can also use Proposition 2 to demonstrate non-planarity of two important graphs.

Theorem 3 *The graphs K_5 and $K_{3,3}$ are not planar.*

Proof. The graph K_5 has 10 edges and 5 vertices, and $10 > 3 \cdot 5 - 6 = 9$. So, by point 1) of Proposition 2 the graph cannot be planar.

The bipartite graph $K_{3,3}$ has 9 edges and 6 vertices, and $9 > 2 \cdot 6 - 4 = 8$. So, by point 2) of Proposition 2 the graph cannot be planar. ■

Both K_5 and $K_{3,3}$ play a prominent role in the theory of planar graphs. They are, as shown in the next section, the epitomes of non-planarity.

2 Subdivisions

Let $G = (V, E)$ be a graph. A *subdivision* of G is, informally, any graph obtained from G by "subdividing" some of its edges by drawing new vertices on those edges. In other words, we replace some edges in E with new paths between their ends, so that none of these new paths has an inner vertex in V or on another new path.

When H is a subdivision of G, we also write that $H = T(G)$. The original vertices of G are called *branch vertices* of $T(G)$; its new vertices are called *subdividing vertices*.

If a graph G' contains a $T(G)$ as a subgraph, then G is called a *topological minor* of G'.

Example 7 Let G, $H = T(G)$ and G' be as in Figure 9. Since $H \subseteq G'$, we have that G is a topological minor of G'.

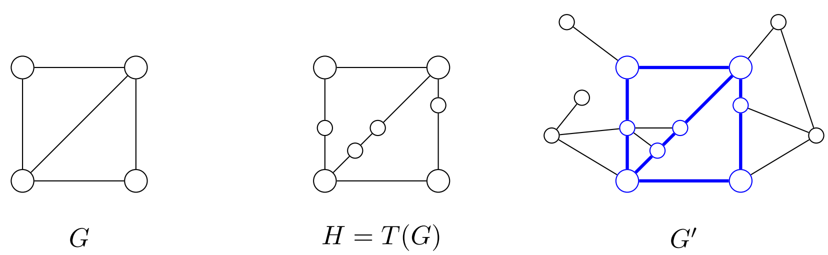


Figure 9: Topological minors.

Clearly, if G is a planar graph, then every subgraph H of G is also planar.

Similarly, if G is a non-planar graph, then any subdivision $T(G)$ is also non-planar.

Theorem 4 (Kuratowski (1930)) *Let G be a graph. Then G is planar if and only if G has neither K_5 nor $K_{3,3}$ as a topological minor.*

The "only if" part of the theorem follows easily from the previous observation and from the already established non-planarity of K_5 and $K_{3,3}$.

The "if" part of the proof of the theorem is quite long and technical.

It may be quite non-intuitive how to use Theorem 4 do demonstrate non-planarity of a particular graph.

Example 8 Let us consider the graph G on the left of Figure 10. The subgraph H of G shown in the centre of the same figure is a topological minor of $K_{3,3}$ (right of the same figure). Therefore H is not planar, and hence G is not planar neither.

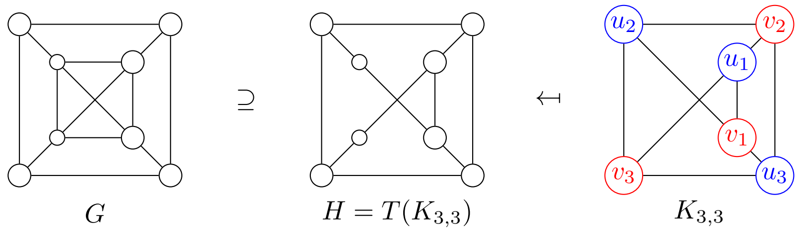


Figure 10: Topological minors.

3 Planarity algorithm

Graph planarity plays a crucial role in many real-world applications where edge crossings can complicate design and functionality. For example:

- **Printed circuit board** (**PCB**) design, where minimizing crossings reduces complexity and improves manufacturability.

- **Subway and transportation network planning**, where clear, non-overlapping routes enhance usability and efficiency.

Determining whether a graph is planar – and, if so, constructing a planar drawing – is a fundamental challenge with broad practical implications.

We present a human-readable explanation of the classical *path addition method* introduced by Hopcroft and Tarjan in 1974. This method was groundbreaking since it was the first linear-time algorithm (with respect to the number of vertices) for testing graph planarity.

Let H be a planar subgraph of G, and let $\hat{H}$ be a planar embedding (i.e., a planar drawing) of H.

We say that $\hat{H}$ is G-*admissible* if G is planar and there exists a planar embedding $\hat{G}$ of G such that $\hat{H}$ is a subgraph of $\hat{G}$.

Example 9 Let us consider the graph G on the left of Figure 11 and its subgraph $H = G \setminus \{\mathtt{a}, \mathtt{f}\}$. The planar embedding $\hat{H}_1$ of H, shown in the centre of the figure, is G-admissible, while the planar embedding $\hat{H}_2$ of H, shown on the right, is not.

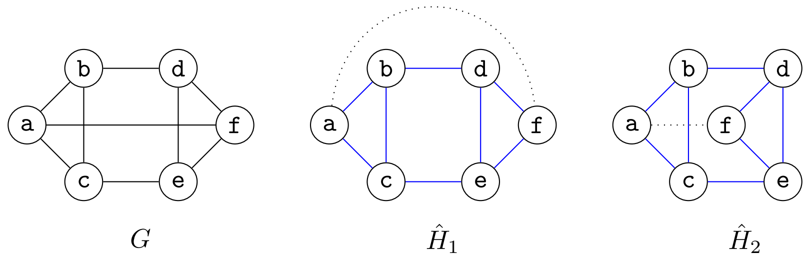


Figure 11: A planar graph G and two embeddings of a subgraph H.

Suppose $H = (V', E') \subset G = (V, E)$. We define an equivalence relation $\sim_B$ on the edges of G that are not in H, i.e., on $E \setminus E'$, as follows: two edges e_1 and e_2 are equivalent, denoted $e_1 \sim_B e_2$, if there exists a walk W in G such that

- the first and last edges of W are e_1 and e_2, respectively;
- W is internally disjoint from H, i.e., no internal vertex of W belongs to H.

The relation $\sim_B$ is an equivalence relation. A subgraph of $G \setminus E'$ induced by an equivalence class under $\sim_B$ is called a *bridge* of H in G.

Example 10 Let us consider the graph G in Figure 12 and its subgraph H given by the cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9)$. Five possible bridges of H in G are shown in the picture.

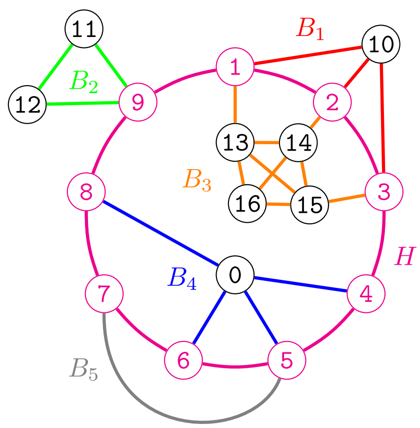


Figure 12: A graph G with a subgraph H and 5 bridges.

Given a planar drawing $\hat{H}$ of H, a bridge B of H is *drawable* in a face f of $\hat{H}$ if the vertices of attachment of B to H lie in the boundary of f.

We denote by $F(B, \hat{H})$ the set of faces of $\hat{H}$ in which B is drawable.

Example 11 Let G be the graph in Figure 13. Let H be the subgraph of G, whose planar drawing $\hat{H}$ is represented in the figure in blue. $\hat{H}$ has tree faces: Ω_1 (the outer face), Ω_2 and Ω_3. The bridge B, with edges represented in red, of H is drawable in Ω_1 and Ω_2, but not in Ω_3, i.e., $F(B, \hat{H}) = \{\Omega_1, \Omega_2\}$.

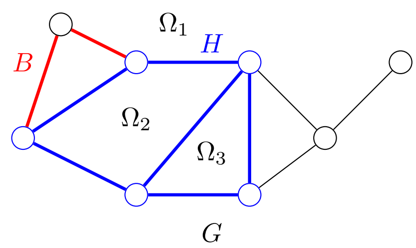


Figure 13: A graph G with a subgraph H and a bridge B.

Theorem 5 *If $\hat{H}$ is G-admissible, then for every bridge B of H in G we have $F(B, \hat{H}) \neq \emptyset$.*

Algorithm

The algorithm constructs an **increasing sequence** of planar subgraphs of $G = (V, E)$:

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots ,$$

with each $G_i = (V_i, E_i)$, along with their corresponding planar embeddings:

$$\hat{G}_1 \subseteq \hat{G}_2 \subseteq \hat{G}_3 \subseteq \cdots .$$

If G is planar, each $\hat{G}_i$ is G-admissible, and the sequence $(\hat{G}_i)_{i\geq 1}$ terminates in a planar embedding of G.

Steps.

1. Initialise G_1 as a cycle in G, find its planar embedding $\hat{G}_1$, and set $i = 1$.

2. If $E \setminus E_i = \emptyset$, then `STOP`: $\hat{G}_i$ is a planar embedding of G.

   Otherwise, determine all bridges of G_i in G. For each such bridge B find the set of faces $F(B, \hat{G}_i)$ in which B is drawable.

3. If there exists a bridge B such that $F(B, \hat{G}_i) = \emptyset$, then `STOP`: G is not planar.

   If there exists a bridge B such that $\#F(B, \hat{G}_i) = 1$, let $\{f\} = F(B, \hat{G}_i)$.

   Otherwise, choose any bridge B and any face $f \in F(B, \hat{G}_i)$.

4. Select a path $P_i \subseteq B$ connecting two vertices of attachment of B to G_i.

   Set $G_{i+1} = G_i \cup P_i$ and obtain $\hat{G}_{i+1}$ by drawing P_i in the face f of $\hat{G}_i$.

   Increment i by 1 and return to Step 2.

Exercise. Let G be the graph shown in Figure 14. Apply the previous algorithm to find a planar embedding of G.

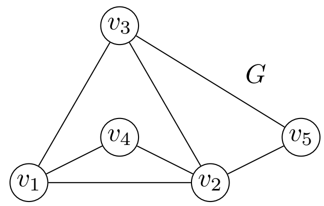
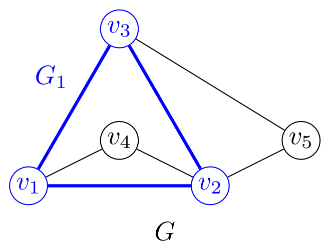


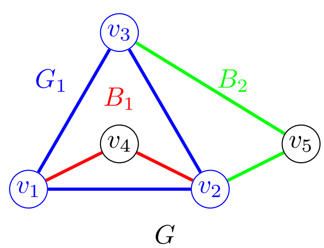
Figure 14: A planar graph.

Solution of Exercise in previous Lecture. Consider the subgraph G_1 of G given by the cycle (v_1, v_2, v_3).

- **Step 1:** G_1 is the cycle (v_1, v_2, v_3), and $\hat{G}_1$ is its planar embedding.

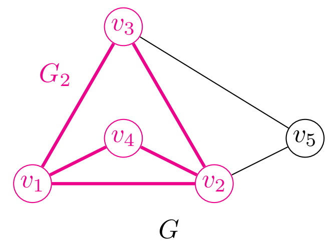


- **Step 2:** The edges (v_1, v_4), (v_4, v_2), (v_2, v_5), and (v_5, v_3) are not in G_1. The bridges of G_1 in G are:
  - B_1 formed by $\{v_1, v_4\}$ and $\{v_4, v_2\}$,
  - B_2 formed by $\{v_2, v_5\}$ and $\{v_5, v_3\}$.



- **Step 3:** Both B_1 and B_2 are drawable. Let us pick the bridge B_1 and the inner face of $\hat{G}_1$.

- **Step 4:** Select the path $P_1 = (v_1, v_4, v_2)$ in B_1. Add P_1 to G_1 to form G_2, and draw P_1 in the inner face of $\hat{G}_1$.



- **Iteration:** Repeat the process for G_2 and the remaining edges until all edges are embedded or non-planarity is detected (in this example the graph is planar).

4 Four Colours Theorem

If any result in graph theory has a claim to be known to the world outside, it is the following Four Colour Theorem.

Theorem 6 (Four Colour Theorem) *Every planar graph has chromatic number at most* 4*.*

The above theorem (proved in 1976 by Appel and Haken) proves the long standing Four Colour Conjecture that every planar graph is 4-colourable. This problem has an easy to understand reformulation "outside" of graph theory:

"*Any map can be coloured using at most four colours in such a way that adjacent regions (i.e., those sharing a common boundary segment, not just a point) receive different colours*"

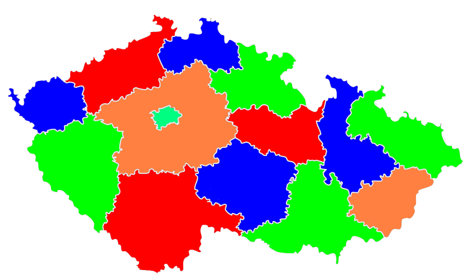


Figure 15: Map of Czechia's regions using only four colours.

Indeed, given a planar graph G one can construct a dual graph G' putting in bijection the set of faces of G with the set of vertices of G', with two faces adjacent in G if and only if the two correspective vertices in G' are connected by an edge.

Example 12 The graph G on the left of Figure 16 is the dual of G' represented on the right of the same figure.

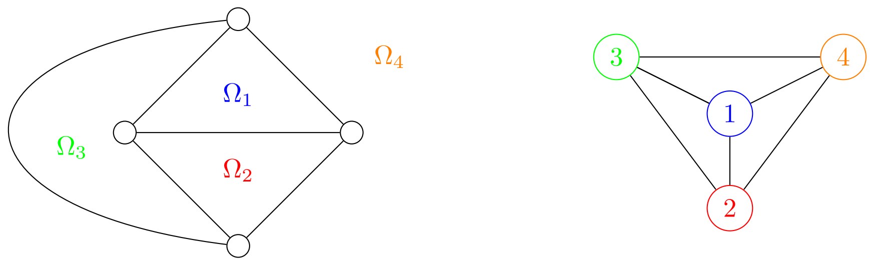


Figure 16: Two dual graphs.

History. The Four Colour Conjecture was first formulated by Francis Guthrie in 1852 while attempting to colour a map of the counties of England. Francis shared the problem with his brother Frederick Guthrie, then an undergraduate at Cambridge, who passed it on to Augustus De Morgan. The conjecture gained broader attention when Arthur Cayley presented it to the London Mathematical Society in 1878.

In 1879, Alfred Kempe published what was initially believed to be a proof, but it was later revealed to be incorrect. However, in 1890, Percy Heawood adapted Kempe's work to prove the Five Colour Theorem, which states that every planar graph G satisfies $\chi(G) \leq 5$.

The first widely accepted proof of the Four Colour Theorem (FCT) was published by Kenneth Appel and Wolfgang Haken in 1977. Their proof followed a two-step approach:

1. **Unavoidable Configurations**: They demonstrated that every planar triangulation must contain at least one of 1,482 specific "unavoidable configurations."

   Reducibility: Using a computer, they showed that each of these configurations is "reducible" – meaning any planar triangulation containing such a configuration can be 4-coloured by combining 4-colourings of smaller triangulations.

Together, these steps formed an inductive proof that all planar graphs can be 4-coloured.

The proof sparked significant controversy, not only due to its reliance on computer assistance but also because of its complexity. In response to criticism, Appel and Haken published a 741-pages algorithmic version in 1989, addressing errors and clarifying the proof. A more concise proof, though still computer-assisted, was later provided by Neil Robertson and colleagues in 1997.

To this day, no "simple" or purely theoretical proof of the Four Colour Theorem has been discovered.