Mathematics for Informatics

Introductory Lecture - Algebraic structures (lecture 1 of 12)

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Organization

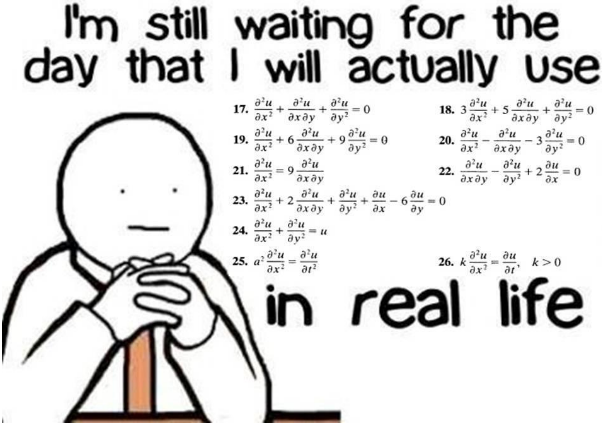
Lecturer:

1. Francesco Dolce (dolcefra@fit.cvut.cz)

Conditions, materials, schedules: https://courses.fit.cvut.cz/NIE-MPI/

see here the conditions to pass the course

Why mathematics?



Why should we learn mathematics?

If someone can take up this position (painlessly), what do you say to yourself?

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If someone can take up this position (painlessly), what do you say to yourself?

Good! I'd like to be agile as she is . . .

OR

Hmm, I didn't need such a daredevil position in my life, I am going to train sitting on a chair instead, that's what I do . . .

Understanding

MATHEMATICS
is not about
numbers, equations,
computations, or
algorithms:
it is about
UNDERSTANDING.

William Paul Thurston

15 Majors that Will Make You Rich (measured by money)

1. Petroleum Engineering ($155,000 – after some time)
2. Physics ($101,800)
3. Applied Mathematics ($98,600 “Jobs in this field can be found in nearly every sector.”)
4. Computer Science ($97,900)
5. Biomedical Engineering ($97,800)
6. Statistics ($93,800)
7. Civil Engineering ($90,200)
8. Mathematics ($89,900)
9. Environmental Engineering ($88,600)
10. Software Engineering ($87,800)
11. Finance ($87,300)
12. Construction Management ($85,200)
13. Biochemistry ($84,700)
14. Geology ($83,300)
15. Management Information Systems ($82,200)

source: `http://likes.com/misc/15-majors-that-will-make-you-rich`

Famous names ...

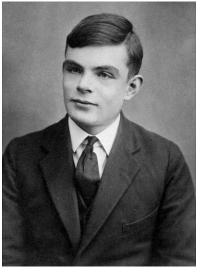
George STIBITZ (Ph.D. in mathematical physics)

He was a Bell Labs researcher known for his work in the 1930s and 1940s on the realization of Boolean logic digital circuits using electromechanical relays as the switching element.

Famous names . . .

Marian Rejewski, Alan Turing, . . . (mathematicians)

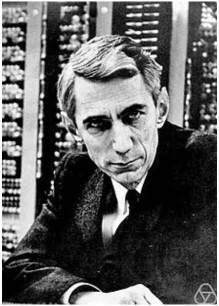
Breaking of German codes during WWII.



Famous names . . .

Claude Shannon, (founder of information theory, mathematician)

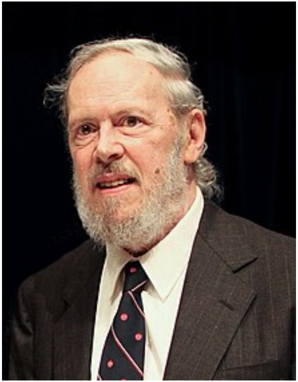
Shannon is famous for having founded information theory with one landmark paper published in 1948. But he is also credited with founding both digital computer and digital circuit design theory in 1937, when, as a 21-year-old master's student at MIT, he wrote a thesis demonstrating that electrical application of Boolean algebra could construct and resolve any logical, numerical relationship.



Famous names . . .

Dennis RITCHIE, (computer scientist, creator of C programming language)

Ritchie graduated from Harvard University with degrees in physics and applied mathematics.



Famous names . . .

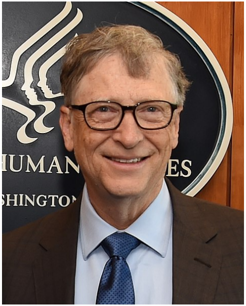
Linus Torvalds (developer of the Linux kernel)

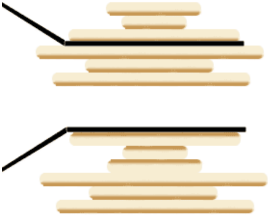
> *His parents were both journalists. However, he was highly influenced by his maternal grandfather to pursue his career in computers. Since childhood, Linus was brilliant in mathematics. In 1988 he began studing computer science at the University of Helsinki. Linus is from a minority group in Finland and his first language is not Finnish but Swedish. For this reason, his pronunciation of Linux in Swedish were not understood or often taken as an error.*

Famous names . . .

Bill GATES (founder of Microsoft)

> *In his sophomore year, Gates devised an algorithm for pancake sorting as a solution to one of a series of unsolved problems presented in a combinatorics class by Harry Lewis, one of his professors. Gates' solution held the record as the fastest version for over thirty years; its successor is faster by only 2%. His solution was later formalized in a published paper in collaboration with Harvard computer scientist Christos Papadimitriou.*





Famous names . . .

Larry Page and Sergey Brin (founders of Google)

Larry was in search of a dissertation theme for his PhD in computer science and considered exploring the mathematical properties of the World Wide Web, understanding its link structure as a huge graph.
After graduation at the University of Maryland, Sergey moved to Stanford University to acquire a Ph.D in computer science.
The company was founded while they were both attending Stanford University.

What about us?

What will we be talking about in this course?

General algebra

Notions from general algebra are one of the basic mathematical tools.

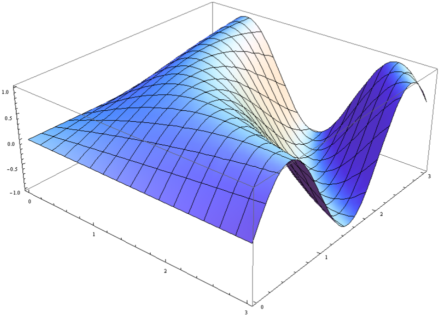
$\cdot$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	1	3	5	7	9	11
3	3	6	9	12	2	5	8	11	1	4	7	10
4	4	8	12	3	7	11	2	6	10	1	5	9
5	5	10	2	7	12	4	9	1	6	11	3	8
6	6	12	5	11	4	10	3	9	2	8	1	7
7	7	1	8	2	9	3	10	4	11	5	12	6
8	8	3	11	6	1	9	4	12	7	2	10	5
9	9	5	1	10	6	2	11	7	3	12	8	4
10	10	7	4	1	11	8	5	2	12	9	6	3
11	11	9	7	5	3	1	12	10	8	6	4	2
12	12	11	10	9	8	7	6	5	4	3	2	1

Cayley table of the group $\mathbb{Z}_{13}^{\times}$

Besides a general introduction, we will focus on finite groups and fields, which form the basis for cryptography, hash functions, etc.

Multivariate functions and optimization

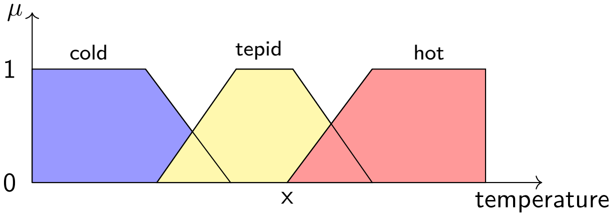
- Many problems can be formulated as optimization problems: we maximize/minimize some functions that determines gain/cost/time/distance ...
- If the function is given analytically, we know how to find the optimum.



$\sin(x \cdot y)$

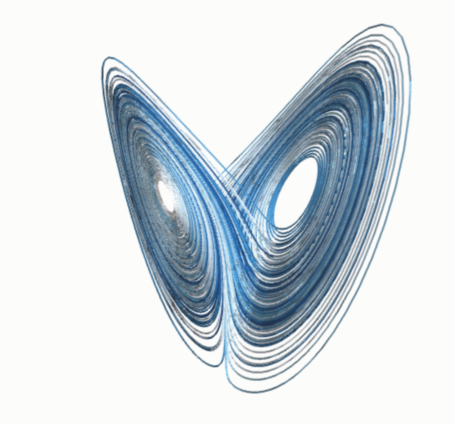
Fuzzy Logic

Describe systems by properties which are not evaluated by values beyond just `true` or `false`.



Numerical mathematics

Continuous mathematics using the computer, stability of numerical algorithms ...



Shall we start?

Outline

- Introduction and motivation
- Hierarchy of sets with one binary operation
  - Introduction
  - Definitions and elementary properties
  - Cayley table
  - Cayley graph

Searching for hidden similarities...

Let us consider this objects:

- the set $\mathbb{Z}$ of integers with the usual sum;
- the set of matrices $\mathbb{R}^{n,n}$ with the operation of matrix multiplication;
- the set of relations on a set A with the operation of relation composition;
- the set $\{0, 1, 2, 3\}$ with the multiplication $\pmod 4$;
- the set of finite automata with the operation of composition;
- the set of all colors with the operation "mixing";
- ...

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All presented objects have the same structure. Indeed, they consist of two ingredients:

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- A (finite or infinite) **set of objects**.
- A **binary operation** mapping two objects onto (exactly) one object (from the same set of objects).

Generally, we speak about a pair of: a **set** and a **binary operation** on it.

We will (mostly) use one of the following notations: $(M, \cdot)$ (multiplicative notation), $(M, +)$ (additive notation), or $(M, \circ)$ (general notation), where

- $M \neq \emptyset$ is a non-empty set, and
- for binary operation we have $\cdot : M \times M \to M$ (resp. $+ : M \times M \to M$, resp. $\circ : M \times M \to M$).

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If we prove some statement for a general structure $(M, \cdot)$*, where* $\cdot$ *is an associative operation, this statement is proved for all particular structures with an associative binary operation!*

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We can understand a general structure as a ***parent object****, from which particular structures* ***inherit*** *all its properties (see below).*

Example of "inheritance" (1/4)

On the set of non-zero real numbers we prove the following (trivial) theorem:

Theorem

For all $b, c \in \mathbb{R} \setminus \{0\}$*, the equation* $bx = c$ *has solution* $x = b^{-1}c$*.*

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What was fundamental for the proof: associativity, existence of (left) inverse element, existence of the neutral element.

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- Is there an inverse matrix for all $A \in M$?
  NO! We have to restrict ourselves to the set of regular matrices M_{reg}.

Example of "inheritance" (3/4)

We have everything needed to prove the theorem for matrices.

Theorem

For all $B, C \in M_{reg}$, the equation $BX = C$ has solution $X = B^{-1}C$.

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Example of "inheritance" (4/4)

Suppose that we are given a pair $(M, \circ)$ where the associativity law holds, for each element $b \in M$ there exists an inverse element, denoted by b^{-1}, and there exists a neutral element e. We will call such pair a group.

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Suppose that we are given a pair $(M, \circ)$ where the associativity law holds, for each element $b \in M$ there exists an inverse element, denoted by b^{-1}, and there exists a neutral element e. We will call such pair a group.
We have a general theorem.

Theorem

For arbitrary elements b, c of a group $(M, \circ)$, the equation $b \circ x = c$ has solution $x = b^{-1} \circ c$.

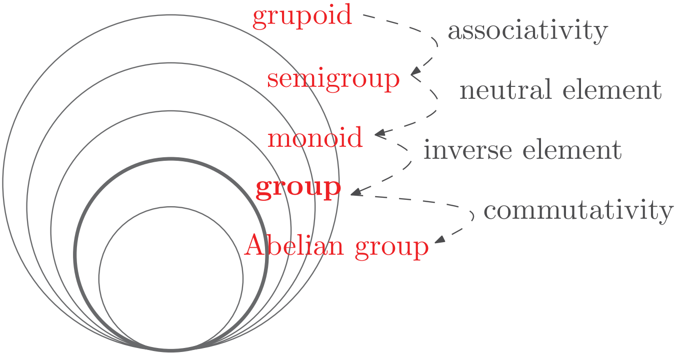
Proof.

$$\begin{aligned} b \circ x &= c && \text{[multiplication on the left by the inverse element } b^{-1}\text{]} \\ b^{-1} \circ (b \circ x) &= b^{-1} \circ c && \text{[moving brackets due to associativity]} \\ (b^{-1} \circ b) \circ x &= b^{-1} \circ c && \text{[for arbitrary } b \text{ we have } b^{-1} \circ b = e\text{]} \\ e \circ x &= b^{-1} \circ c && \text{[for arbitrary } x \text{ we have } e \circ x = x\text{]} \\ x &= b^{-1} \circ c \end{aligned}$$

□

Sets with one binary operation

We call an arbitrary pair "a set and a binary operation" a groupoid. Adding another requirements we get further notions.



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  It is an Abelian group.
- For the pair $(\mathbb{Z}, +)$ associative and commutative laws hold, the neutral element is 0 and the inverse element for b is $b^{-1} = -b$.
  It is an Abelian group.
- For the pair $(M_{\text{reg}}, \cdot)$ associativity law holds, the neutral element and the inverse exist, but the commutative law is not valid!
  It is a group, but not Abelian.

Mathematical analogy to Object-oriented programming

We can consider the groupoid, monoid, etc., as mathematical (abstract) objects, for which a nonempty set and a binary operation with given properties are defined.

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This analogy could be employed in real programming.

Groupoid, semigroup, monoid, group

Definition

- *An ordered pair* $(M, \circ)$, *where* M *is an arbitrary non-empty set and* $\circ$ *is a binary operation on* M, *is called a groupoid.*

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- *An ordered pair $(M, \circ)$, where M is an arbitrary non-empty set and $\circ$ is a binary operation on M, is called a groupoid.*
- *A groupoid $(M, \circ)$ such that $\circ$ is associative is called a semigroup.*
- *A semigroup $(M, \circ)$ such that there exists a neutral element e satisfying*

$$\forall\, a \in M \quad \textit{holds} \quad e \circ a = a \circ e = a$$

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Groupoid, semigroup, monoid, group

Definition

- *An ordered pair* $(M, \circ)$, *where* M *is an arbitrary non-empty set and* $\circ$ *is a binary operation on* M, *is called a groupoid.*
- *A groupoid* $(M, \circ)$ *such that* $\circ$ *is associative is called a semigroup.*
- *A semigroup* $(M, \circ)$ *such that there exists a neutral element* e *satisfying*

$$\forall\, a \in M \quad \textit{holds} \quad e \circ a = a \circ e = a$$

is called a monoid.

- *A monoid* $(M, \circ)$ *such that for each* $a \in M$ *there exists an inverse element* $a^{-1} \in M$ *satisfying*

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  is called a group.
- *Moreover, if* $\circ$ *is commutative, we say that a group* $(M, \circ)$ *is a commutative (or Abelian) group.*

Set closed under the binary operation. What does it mean?

In the definition we require the binary operation $\circ$ to be a "binary operation on M".

This means that the result of a binary operation applied on two elements from M again belongs to M – we say that the **set M is closed under $\circ$.**

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Example

The pair $(\mathbb{Z}_-, \cdot)$ *of negative integers with the usual multiplication is not a groupoid, because it is not closed under the operation:* $(-1) \cdot (-1) = 1 \notin \mathbb{Z}_-$.

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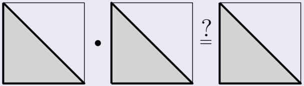
Whether the set is/is not closed under the binary operation is not always obvious.

Example

Let us consider the couple $(M_{triang}, \cdot)$ of lower triangular matrixes with the usual matrix multiplication. Is M_{triang} closed under the operation $\cdot$?



Manual for classification of sets with binary operation

If we have a given pair "a set and a binary operation" and we want to find out whether it is a groupoid, semigroup, monoid, (Abelian) group, we can proceed this way:

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Mostly "proofs" in these individual steps are very easy or obvious. Sometimes, they only *seem* obvious.

Groupoid, semigroup, monoid, group – examples (1/4)

Example

Let us consider the groupoid $(\mathbb{Q}, \circ)$, *where the binary operation* $\circ$ *is defined as the arithmetic mean:*

$$a \circ b := \frac{a+b}{2}.$$

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So, the associative law does not hold, and the structure is not a semigroup. It follows that $\mathbb{Q}$ *with this operation is neither a monoid nor a group.*

Groupoid, semigroup, monoid, group – examples (2/4)

Example

Let us consider a groupoid $(\mathbb{R}^+, \circ)$*, where the binary operation* $\circ$ *is defined as follows:*

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- *Is* $(\mathbb{R}^+, \circ)$ *a monoid?*

Groupoid, semigroup, monoid, group – examples (3/4)

Example

Let us consider a groupoid $(\mathbb{R}, \cdot)$, *where the binary operation is the usual multiplication of numbers.*

- *Is it a semigroup?*
- *Is it a monoid?*
- *Is it a group?*

Groupoid, semigroup, monoid, group – examples (4/4)

From the definition it follows that each group is a monoid, each monoid is a semigroup and each semigroup is a groupoid. Written in symbols we get:

$$\text{groupoids} \supset \text{semigroups} \supset \text{monoids} \supset \text{groups}\,.$$

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From the previous three examples we can be even more specific:

$$\text{groupoids} \supsetneq \text{semigroups} \supsetneq \text{monoids} \supsetneq \text{groups}\ ,$$

because we have found a groupoid that is not a semigroup, a semigroup that is not a monoid, and a monoid that is not a group.

Uniqueness of neutral element

Theorem

Given a monoid, there exists exactly one neutral element.

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Proof.

Let $(M, \circ)$ be a monoid and e some neutral element (by definition we know that at least one exists!).
We prove by contradiction that e is the only neutral element.
By contradiction, assume that in the monoid there exists another neutral element e' different from e.
Using the property of the neutral element, it holds that

$$e' = e' \circ e = e.$$

We get a contradiction with the assumption that $e' \neq e$. □

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Proof.

Let $(G, \circ)$ be a group, a an arbitrary element of the group and a^{-1} one of its inverse elements (from the definition of a group we know that there exists at least one!).
We prove *by contradiction* that a^{-1} is the only one.
Assume that there exists another inverse element $\overline{a}$ different from a^{-1}. Hence it holds that

$$\overline{a} = \overline{a} \circ e = \overline{a} \circ \left(a \circ a^{-1}\right) = (\overline{a} \circ a) \circ a^{-1} = e \circ a^{-1} = a^{-1}$$

where e is the unique neutral element.
Thus we get a contradiction with the assumption that $\overline{a} \neq a^{-1}$. □

Cayley tables for finite groups

If the set M from the pair $(M, \circ)$ has a finite number of elements, its structure (with the given operation $\circ$) could be completely represented by the Cayley table. Its construction is obvious from the following example.

Example

Let us consider $(\mathbb{Z}_4, +_4)$, *i.e., the set of numbers* $\{0, 1, 2, 3\}$ *with addition modulo* 4.

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$+_4$	0	1	2	3
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1				
2				1
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So, in the cell in row m *and column* n *we write the result of* $m +_4 n = m + n \ (mod\ 4)$.
For example the cell in row 2 *and column* 3 *is filled with* $2 + 3 \ (mod\ 4) = 1$.

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2	2	3	0	1
3	3	0	1	2

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- The inverse element to the element a is the one corresponding to the row and column where the neutral element e is placed.
- ...

Cayley table and latin square (1/4)

Question: Is it possible to recognize whether a table is a Cayley table of a group?
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A latin square for a set M of n elements is a matrix $n \times n$ such that each row and column contains all elements of the set M.

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Unfortunately, not each Cayley table forming a latin square is a Cayley table of a group. Later we present a counterexample.

Cayley table and latin square (2/4)

Theorem

In each group, we can divide uniquely.
In other words: in each group $(G, \circ)$, *for arbitrary* $a, b \in G$ *the equations*

$$a \circ x = b \quad \text{and} \quad y \circ a = b$$

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Proof.

Since we are in a group, each element has only one inverse.
The only solutions of the equations are $x = a^{-1} \circ b$ and $y = b \circ a^{-1}$. □

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It is possible to prove that a group is a semigroup with a "unique division", i.e., the unique division guarantees the existence of a neutral element and inverse.

Cayley table and latin square (3/4)

Now we prove the theorem saying that the Cayley table of group is a latin square.

Proof.

Proof by contradiction.
Let us suppose that the table of some group $(G, \circ)$ is not a latin square.
Hence, in some row or column there is one element, denote it as b, repeated twice. WLOG[a], assume that it happens in row n and columns m_1 and m_2.

$\circ$	$\cdots$	m_1	$\cdots$	m_2	$\cdots$
$\vdots$		$\vdots$		$\vdots$	
n	$\cdots$	b	$\cdots$	b	$\cdots$
$\vdots$		$\vdots$		$\vdots$	

It follows that the equation $n \circ x = b$ has two different solutions, namely m_1 and m_2, which is a **contradiction with the previous theorem**! □

[a]Without Loss Of Generality

Cayley table and latin square (4/4)

We have shown that the fact that a Cayley table is a latin square is a *necessary* condition for the given set and operation to be a group.

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The following example says it is not a *sufficient* condition.

Example

Let us consider a set $M = \{a, b, c\}$ with operation given by the Cayley table:

$\circ$	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

This table creates a latin square; in spite of it, it is not the table of a group (Why?!).

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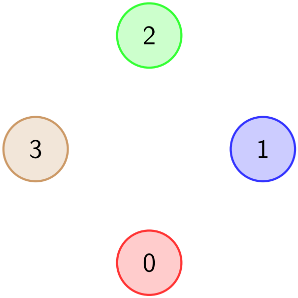

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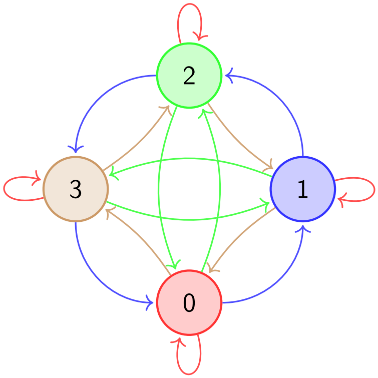

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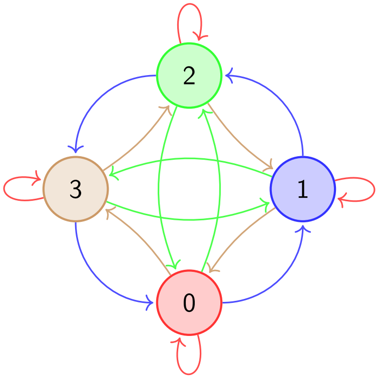

If the group in question is not Abelian, we need to depict edges (a, b) for $b = c \circ a$ for some $c \in M$.