Mathematics for Informatics

Algebra: Subgroups, groups generated by a set, cyclic groups (lecture 2 of 12)

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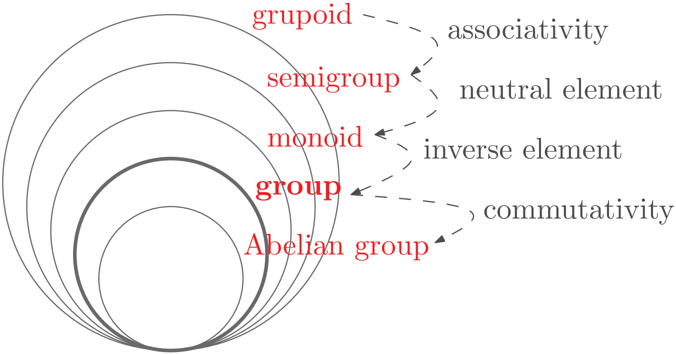
Outline

- Reminder and motivation
- Subgroups
- Groups generated by a set
- Cyclic groups

Reminder of the last lecture

Hierarchy of structures of type “a set and a binary operation”



Example (1/4)

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Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a* ***groupoid***;

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- *the inverse of* $k \neq 0$ *is* $12 - k$ *and the inverse of* 0 *is* 0, *so it is a* ***group****;*

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Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the set of the residue classes modulo n.

The group $(\mathbb{Z}_n, +_{(\mathrm{mod}\ n)})$ is the additive group modulo n; it is denoted by $\mathbb{Z}_n^+$.

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Question (refined): Which subset of $\mathbb{Z}_{12}$ forms a group with the addition $\pmod{12}$?

Answer: There are quite a lot of them. To find out how to discover them, let us ask this subquestion:

Sub-question: Which is the smallest subset of $\mathbb{Z}_{12}$ that forms a group with addition $\pmod{12}$ and contains the number 2?

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- the operation remains associative, so it is a semigroup;
- 0 remains the neutral element, so it is a monoid;
- each element has its inverse in the set (the set is closed under inversion), so it is a group.

The wanted set is $M = \{0, 2, 4, 6, 8, 10\}$.
We say that M is a subgroup generated by the set $\{2\}$.

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Similarly, as we have generated the set from the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

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Back to the original question: there exist 6 different sets $M \subseteq \mathbb{Z}_{12}$ such that $(M, +_{(\text{mod } 12)})$ is a group.

Definition of subgroup

Definition

Let $G = (M, \circ)$ *be a group.*
A subgroup of the group G *is a pair* $H = (N, \circ)$ *such that:*

- $N \subseteq M$ *and* $N \neq \emptyset$,
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- A binary operation in the group $G = (M, \circ)$ is a function from $M \times M$ to M. The operation in a subgroup $H = (N, \circ)$ is, to be precise, the restriction of this operation to the set $N \times N$.

Trivial and proper subgroups

In each group $G = (M, \circ)$, there always exist at least two subgroups (if M contains only one element the two coincide):

- the group containing only the neutral element: $(\{e\}, \circ)$, and
- the group itself $G = (M, \circ)$.

These two groups are the trivial subgroups.
Other subgroups are non-trivial or proper subgroups.

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Question

If H is a subgroup of a group G, is the neutral element of H identical to the neutral element of G?

Intersection of subgroups

Theorem

Let $H_1, H_2, \ldots, H_n$*, whith* $n \geq 1$*, be subgroups of a group* $G = (M, \circ)$*. Then*

$$H' = \bigcap_{i=1,2,\ldots,n} H_i$$

is also a subgroup of G*.*

Power of an element

Definition

Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as

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\begin{aligned}
a^0 &= e \\
a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\
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For all $n, m \in \mathbb{N}$ the following "natural" equalities are true:

- $a^{n+m} = a^n \circ a^m$,
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For the additive notation of a group $G = (M, +)$, we define the n-th multiple of the element a and we denote it by $n \times a$ (resp. $-n \times a = n \times (-a)$).

Order of a (sub)group

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Example

The group $\mathbb{Z}_{12}^{+}$ is of order 12. It has 6 subgroups:

- *two trivial: $\{0\}$ and $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$;*
- *and four proper: $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, and $\{0, 2, 4, 6, 8, 10\}$.*

of order 1, 2, 3, 4, 6 and 12.

(Left) cosets of a subgroup

Let G be a group and H be one of its subgroups.
The (left) coset of H in G with respect to an element $g \in G$ is the set

$$gH = \{gh : h \in H\} \quad \text{(or } g + H \text{ in additive notation)}$$

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Let us consider the subgroup $H = \{0, 3, 6, 9\}$ of $\mathbb{Z}_{12}$.
Find $g + H$ for all $g \in \mathbb{Z}_{12}$.

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The index of H in G, denoted $[G : H]$, is the number of different cosets of H in G.

Lagrange's Theorem

Theorem

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This statement connects the abstract structure of a group with divisibility and also with the notion of prime numbers!

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To prove Lagrange's Theorem we need the following lemma.

Lemma

For all $a, b \in G$ one has $|aH| = |bH|$.

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Lemma

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Question

*Let G be a group of order n and $k \in \mathbb{N}$ be such that $k|n$.
Is there any subgroup of G of order k?*

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Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The smallest subgroup of G containing N is the subgroup generated by N and is denoted by $\langle N \rangle$.

In particular, for a singleton $N = \{a\}$ we use the notation $\langle a \rangle = \langle \{a\} \rangle$.

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Definition

If for a set M it holds that $\langle M \rangle = G$, we say that M is a generating set of G.

Group generated by a set (2/2)

Example

The group $\mathbb{Z}_{12}^{+}$ *is generated, for instance, by the sets* $\{1\}$ *and* $\{5\}$*, i.e.*

$$\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_{12}^{+}.$$

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- *all elements in $\langle N \rangle$ can be obtained by means of "group span", i.e.,*

$$\left\{ a_1^{k_1} \circ a_2^{k_2} \circ \cdots a_n^{k_n} \ : \ n \in \mathbb{N}, \ a_i \in N, \ k_i \in \mathbb{Z} \right\}.$$

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Groups generated by one element (1/2)

We have seen that the additive group $\mathbb{Z}_{12}^{+}$ is equal to $\langle 1 \rangle$, $\langle 5 \rangle$, $\langle 7 \rangle$, and $\langle 11 \rangle$.

The following theorem generalize this fact.

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Proof.

This statement is a consequence of a general theorem which will be proven later and of the fact that $\mathbb{Z}_{n}^{+} = \langle 1 \rangle$ for all $n \geq 2$. $\square$

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The group $(\{1, 2, \ldots, p-1\}, \cdot_{(\text{mod } p)})$, where p is a prime number, is the multiplicative group modulo p, denoted $\mathbb{Z}_p^\times$.

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Finding the generator(s) of a multiplicative group $\mathbb{Z}_p^\times$ is more complicated than for an additive group $\mathbb{Z}_n^+$.
Multiplicative groups have more complicated and interesting structure.

Definition of cyclic group

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- The infinite group $(\mathbb{Z}, +)$ is cyclic and it has just two generators: 1 and -1.
- $\mathbb{Z}_{11}^\times$ is cyclic, and 2 is a generator.

Why "cyclic"?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

If we repeatedly compose the generator 2 with itself we successively get all elements of the group: $2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 3, \ldots, \quad 2^{12} = 1$.
The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

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Fermat's Theorem (1/2)

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In a cyclic group $G = (M, \circ)$ of order n, for all elements $a \in M$, it holds that

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Where e is the neutral element of G.

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We have $a^n = a^{qk} = (a^q)^k = e^k = e$. $\square$

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Applying the previous theorem to $\mathbb{Z}_p^\times$ we obtain the well-known Fermat's Little Theorem.

Corollary (Fermat's Little Theorem)

For an arbitrary prime number p and an arbitrary $1 \leq a < p$ we have that

$$a^{p-1} \equiv 1 \,(mod\ p).$$

How to find all generators (1/2)

Generally, to find all generators is not an easy task (e.g., in groups $\mathbb{Z}_p^\times$ we are not able to do it algorithmically); but if we have one, it is easy to find all the others.

Theorem

If $(G, \circ)$ is a cyclic group of order n and a is one of its generator, then a^k is a generator if and only if k and n are coprime.

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To prove the previous theorem we use the following

Lemma

Let $D = \{mk + \ell n \mid m, \ell \in \mathbb{Z}\}$.
Then $\gcd(k, n) = \min\{|a| \mid a \in D \setminus \{0\}\}$.

How to find all generators (2/2)

Corollary

In a cyclic group of order n, the number of all generators is equal to $\varphi(n)$.

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An effective algorithm for evaluating $\varphi(n)$ does not exist; if it existed, we would be able to find the integer factorization of arbitrarily large n and RSA would not be safe!

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Example

Find the order of all elements in $\mathbb{Z}_5^\times$ *and in* $\mathbb{Z}_7^\times$.