BIE-PST – Probability and Statistics

Lecture 11: Hypothesis testing

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Lecturer:
Francesco Dolce

Department of Applied Mathematics
Faculty of Information Technology
Czech Technical University in Prague

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11 Hypothesis testing

11.1 General theory

Often we need to verify a claim about some property of a studied distribution, with only a random sample at our disposal.

Delivery of goods We accept a delivery of goods if the percentage of faulty items is less then some given boundary, e.g., 5%.

Often it is not possible to check the quality of all items. Therefore we select a random sample and test just the quality of sampled items. On the basis of the result we need to decide whether to accept the delivery or not.

Improving a technological procedure A new technological procedure has been invented. Before using it in normal operation we need to decide whether the new procedure is actually better than the old one or not.

We measure samples of n_1 and n_2 products manufactured using the old and the new procedure. On the basis of these samples we need to decide whether there is a difference between the old and the new method or not.

Let us observe a *random sample* from some distribution. *Statements about* this *distribution*, which cannot be surely confirmed, are called *hypotheses*.

The mechanism of verifying the validity of hypotheses based on observed data is called *hypothesis testing*.

We use two *basic notions*:

- *Null hypothesis* H_0 denoting the statement which we want to confirm or reject.
- *Alternative hypothesis* H_A is a converse statement against which we reject H_0.

Hypotheses types:

- *Non-parametric* – random sample from a general distribution. The statements deal with various properties of the distribution (e.g., median), or the shape of the distribution (goodness-of-fit tests).
- *Parametric* – random sample from a distribution given by parameters $\theta \in \mathbb{R}^d$. We test statements regarding the value of θ.

The *test* of a given null hypothesis H_0 against an alternative hypothesis H_A is a *decision procedure* with two possible results: either *rejecting* or *not rejecting* the null hypothesis H_0.

While deciding we can make one of two *errors*:

- Rejecting H_0 even if it is valid – *type I error*.
- Not rejecting H_0 even if it is not valid (if H_A is valid) – *type II error*.

Often it is possible to have only one of these errors under control.

- We proceed so that the probability of making the *type I error* is *less than or equal* to some small given α, which is called the *level of significance*.
- Usually we take $\alpha = 5\%$ or $\alpha = 1\%$.
- The type II error can be either small or large depending on the sample size.
- The probability of *not making* type II error is called the *power* of the test.

Possible results of testing:

- *We reject* the hypothesis H_0 in favor of H_A, with a small probability α of making an error.
- *We do not reject* H_0.

The position of both hypotheses is not symmetric.

As a null hypothesis we choose that one, where the wrongful rejection, i.e., making type I error, would be more serious.

The hypothesis which we need to confirm is chosen as the alternative hypothesis H_A.

Rejection of H_0 in favor of H_A is a strong result.
Generally we say that we "test a hypothesis H_0 against an alternative H_A at the level of significance α".

- If we reject a hypothesis H_0 and thus can very reliable state that the alternative H_A holds, we say that the statement of H_A is *"statistically significant"*.
- If we do not reject H_0, the statement of H_A is called *"statistically insignificant"*.

11.2 Parametric tests

Let $X_1, \ldots, X_n$ be a random sample from a distribution with a parameter θ.

We want to test the hypothesis (*two-sided alternative*):

$$H_0 : \theta = \theta_0 \qquad \text{against} \qquad H_A : \theta \neq \theta_0,$$

for some fixed value θ_0.

Let further $\big(L(\mathbf{X}), U(\mathbf{X})\big)$ be the *two-sided* $100(1-\alpha)\%$ *confidence interval* for θ based on a random sample. Thus it holds that

$$\mathrm{P}\left(\theta \in (L, U)\right) = 1 - \alpha.$$

We decide as follows:

- *We reject* the hypothesis H_0 if $\theta_0 \notin (L, U)$.

- *We do not reject* H_0 if $\theta_0 \in (L, U)$.

We verify that this way we can test the hypothesis at given *significance level* α.

If the null hypothesis H_0 holds, i.e., $\theta = \theta_0$, for the *type I error* we have

$$\begin{aligned}\mathrm{P}(\text{reject } H_0 | H_0 \text{ holds}) &= \mathrm{P}\left(\theta_0 \notin (L, U) | \theta = \theta_0\right) = \mathrm{P}\left(\theta \notin (L, U)\right) \\ &= 1 - \mathrm{P}\left(\theta \in (L, U)\right) = 1 - (1 - \alpha) = \alpha,\end{aligned}$$

because (L, U) is the $100(1-\alpha)\%$ confidence interval for θ. The *level of significance* of our test is indeed α. There are more possible decision rules. However, it can be shown (see literature) that for a general class of distributions, using the $(1-\alpha)$ confidence intervals, the probability of making type II error is lowest for any test with level of significance α. Therefore we can obtain the *most powerful test* against the given alternative.

Now we want to test the hypothesis against a *one-sided alternative*

$$H_0 : \theta = \theta_0 \qquad \text{against} \qquad H_A : \theta > \theta_0.$$

For testing we use the one-sided interval of the type *corresponding to the alternative hypothesis*, i.e., *upper* $100(1-\alpha)\%$ *confidence interval* $(L, +\infty)$ for θ. It holds that

$$\mathrm{P}\left(\theta \in (L, +\infty)\right) = \mathrm{P}(\theta > L) = 1 - \alpha.$$

We decide as follows:

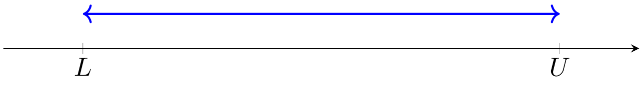
- *We reject* the hypothesis H_0 if $\theta_0 \notin (L, +\infty)$, i.e., $\theta_0 < L$.
- *We do not reject* the hypothesis H_0 if $\theta_0 \in (L, +\infty)$, i.e., $\theta_0 \geq L$.

Remarks:

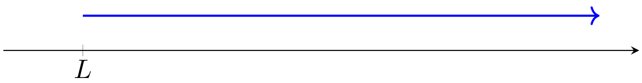
- If θ_0 is outside the interval, the alternative H_A holds with a large probability.
- The *level of significance* is again α.
- We proceed analogously for $H_A : \theta < \theta_0$.
- The null hypothesis can also be formulated in a compound form as:

$$H_0 : \theta \leq \theta_0 \qquad \text{against} \qquad H_A : \theta > \theta_0.$$

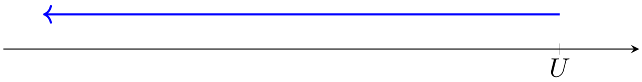
Reject $H_0 : \theta = \theta_0$ in favor of the two-sided alternative $H_A : \theta \neq \theta_0$, if θ_0 does not lie in the *two-sided* confidence interval.



Reject $H_0 : \theta = \theta_0$ in favor of the one-sided alternative $H_A : \theta > \theta_0$, if θ_0 does not lie in the *upper one-sided* confidence interval.



Reject $H_0 : \theta = \theta_0$ in favor of the one-sided alternative $H_A : \theta < \theta_0$, if θ_0 does not lie in the *lower one-sided* confidence interval.



Testing procedure:

- Choose the level of significance α.
- Measure (observe) the random sample.
- Construct a $(1 - \alpha)$ confidence interval corresponding to the alternative hypothesis H_A.
- Reject H_0 if θ_0 is outside of the confidence interval.

11.3 Parametric tests – normal distribution

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

$H_0 : \mu = \mu_0$ against the alternative $H_A : \mu \neq \mu_0$ at the level of significance α:

- For *known* variance σ^2 we reject hypothesis H_0 if μ_0 *is not in* the interval
$$\left(\bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \;,\; \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right).$$
- For *unknown* variance σ^2 we reject hypothesis H_0 if μ_0 *is not in* the interval
$$\left(\bar{X}_n - t_{\alpha/2,n-1} \frac{s_n}{\sqrt{n}} \;,\; \bar{X}_n + t_{\alpha/2,n-1} \frac{s_n}{\sqrt{n}} \right).$$

$H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_A : \sigma^2 \neq \sigma_0^2$ at the level of significance α:

- We reject hypothesis H_0 if σ_0^2 *is not in* the interval
$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha/2,n-1}} \;,\; \frac{(n-1)s_n^2}{\chi^2_{1-\alpha/2,n-1}} \right).$$

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

$H_0 : \mu = \mu_0$ against the alternative $H_A : \mu > \mu_0$ at the level of significance α:

- For *known* variance σ^2 we reject hypothesis H_0 if μ_0 *is not in* the interval
$$\left(\bar{X}_n - z_{\alpha} \frac{\sigma}{\sqrt{n}} \;,\; +\infty \right).$$

- For *unknown* variance σ^2 we reject hypothesis H_0 if μ_0 *is not in* the interval

$$\left(\bar{X}_n - t_{\alpha,n-1}\frac{s_n}{\sqrt{n}}\ ,\ +\infty\right).$$

$H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_A : \sigma^2 > \sigma_0^2$ at the level of significance α:

- We reject the hypothesis H_0 if σ_0^2 *is not in* the interval

$$\left(\frac{(n-1)s_n^2}{\chi^2_{\alpha,n-1}}\ ,\ +\infty\right).$$

Example 11.1. We have $n = 35$ observations of random variable with distribution $\mu = \mathrm{E}\,X$:

$$90\%\text{ interval } A: \quad (0.4055, 5.3945)$$

$$95\%\text{ interval } B: \quad (-0.0724, 5.8724)$$

Test hypothesis

$$H_0 : \mu = 0 \qquad \textit{against} \qquad H_A : \mu > 0$$

at the significance level 5% ($\alpha = 0.05$) and 2.5% ($\alpha = 0.025$).

The needed one-sided confidence interval is

- 5% – $(0.4055, +\infty)$ and because $0 \notin (0.4055, +\infty)$ *we reject* the null hypothesis at significance level $\alpha = 5\%$

- 2.5% – $(-0.0724, +\infty)$ and because $0 \in (-0.0724, +\infty)$ *we cannot reject* the null hypothesis at significance level $\alpha = 2.5\%$.

In bibliography you can encounter the *p-value* approach.

Given observed data, the null hypothesis can not be rejected on every significance level α.

The *minimal significance level* at which we can reject hypothesis H_0 given the data at hand is called the *p-value*. The p-value depends on the random sample realization.

Meaning of the p-value

- Many statistical softwares give only the p-value as the output of a hypothesis test.

- If the p-value is smaller than our required significance level α we reject H_0.

- The size of the p-value informs us how strong is the rejection of H_0 is, or how weak the non-rejection.

- The smaller the p-value is, the more significant is the rejection of H_0.

11.4 Critical regions and tests statistics

The following parts regarding test statistics are not necessary to learn for passing the exam. Some parts may be necessary to solve the homework. However, it is useful to know about this approach, because it is more general than testing based on confidence intervals. The test statistics approach will be used in the master's course MIE-SPI and possibly in further advanced courses. Furthermore, it is often useful in real-life problems involving hypothesis testing.

We can use an other approach for hypotheses testing, based on comparing the tested value with its point estimate. For example, when observing a random sample from $\mathrm{N}(\mu, \sigma^2)$, we could reject the hypothesis $H_0 : \mu = \mu_0$ if the sample mean $\bar{X}_n$ and the tested value μ_0 are far away from each other. How far? We know that if H_0 holds and σ^2 is known, it holds that

$$\frac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n} \sim \mathrm{N}(0, 1).$$

Therefore with a large probability $1 - \alpha$, the standardised distance should be within bounds given by the critical values of $\mathrm{N}(0, 1)$. We can thus reject H_0 in favor of $H_A : \mu \neq \mu_0$ if

$$\left|\frac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n}\right| > z_{\alpha/2}.$$

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

Test $H_0 : \mu = \mu_0$ against the alternative $H_A : \mu \neq \mu_0$ at the significance level α:

If the variance σ^2 is known, we can construct a *test statistic* $T = T(X_1, \ldots, X_n)$ as

$$T = \frac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n}.$$

We reject H_0 in favor of H_A if the test statistic lies in the *critical region*:

$$W_\alpha = \{T : |T| > z_{\alpha/2}\}.$$

We reject the hypothesis H_0 if $T \in W_\alpha$, meaning that $|T|$ is large enough and thus $\bar{X}_n$ is too far from μ_0. Alternatively, we do not reject if it holds that

$$-z_{\alpha/2} < \frac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n} < z_{\alpha/2}.$$

After separating μ_0 we obtain the same interval as the corresponding $(1 - \alpha)$ confidence interval.

The approach based on the construction of the *test statistic* T and the corresponding *critical region of the test* W_α can be summarized as follows:

Testing procedure

- Choose the level of significance α.
- Measure (observe) a random sample.

- Compute the test statistic T.
- Find the corresponding critical region based on the outlying parts of the distribution of T.
- Reject H_0 if $T \in W_\alpha$.

✓ *The critical region can be often converted to the corresponding confidence interval.*

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

Tests for the expectation at significance level α:

- For a *known* variance σ^2 the test statistic and critical regions are following:

H_0	H_A	test statistic T	critical region W_α
$\mu = \mu_0$	$\mu \neq \mu_0$		$\lvert T\rvert > z_{\alpha/2}$
$\mu \leq \mu_0$	$\mu > \mu_0$	$T = \dfrac{\bar{X}_n - \mu_0}{\sigma}\sqrt{n}$	$T > z_\alpha$
$\mu \geq \mu_0$	$\mu < \mu_0$		$T < -z_\alpha$

- For *unknown* variance σ^2 the test statistic and critical regions are following:

H_0	H_A	test statistic T	critical region W_α
$\mu = \mu_0$	$\mu \neq \mu_0$		$\lvert T\rvert > t_{\alpha/2,n-1}$
$\mu \leq \mu_0$	$\mu > \mu_0$	$T = \dfrac{\bar{X}_n - \mu_0}{s_n}\sqrt{n}$	$T > t_{\alpha,n-1}$
$\mu \geq \mu_0$	$\mu < \mu_0$		$T < -t_{\alpha,n-1}$

Tests for the variance at significance level α:

H_0	H_A	test statistic T	critical region W_α
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$		$T > \chi^2_{\alpha/2,n-1} \vee T < \chi_{1-\alpha/2,n-1}$
$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$T = \dfrac{(n-1)s_n^2}{\sigma_0^2}$	$T > \chi^2_{\alpha,n-1}$
$\sigma^2 \geq \sigma_0^2$	$\sigma^2 < \sigma_0^2$		$T < \chi^2_{1-\alpha,n-1}$

11.5 Two-sample and paired tests

Paired t-test

Suppose we observe a random sample of pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$. The variables within pairs can be dependent, but the pairs are independent between each other. Such a situation can describe the value of a certain marker measured on patients before and after a clinical procedure. We want to determine, whether the marker stayed the same, or if it has significantly increased or decreased after the procedure. Suppose that all variables are normally distributed with $X_i \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_i \sim \mathrm{N}(\mu_2, \sigma_2^2)$. We want to test $H_0 : \mu_1 = \mu_2$. When we take $Z_i = Y_i - X_i$, the resulting variables will have the normal distribution with expectation of $\mu_{\text{diff}} = \mu_2 - \mu_1$. The test can then be performed in the same way as for a single sample from a normal distribution, testing $H_0 : \mu_{\text{diff}} = 0$ against $H_A : \mu_{\text{diff}} \neq 0$. Similarly for one-sided alternatives.

Example 11.2 (– comparing fathers' and sons' heights)**.** Suppose we want to determine whether the men's height increases between generations. We have observed five pairs of fathers and their sons, now adults. Their height was measured as follows (in centimeters):

height of father	X_i	172	176	180	184	186
height of son	Y_i	178	188	177	192	193
difference	$Z_i = Y_i - X_i$	6	12	-3	8	7

We test whether the expected sons' height is equal to the expected fathers' height, against the alternative that sons are significantly taller, using $\alpha = 5\%$. The upper one-sided 95% confidence interval for the expectation μ_{diff} of Z_i is

$$\left(\bar{Z}_n - t_{\alpha,n-1}\frac{s_Z}{\sqrt{n}}\ ,\ +\infty\right) = \left(6 - 2.132 \cdot \frac{5.52}{\sqrt{5}}\ ,\ +\infty\right) = (0.735, +\infty).$$

The tested value $\mu_{\text{diff}} = 0$ does not lie in the interval, so we can reject the hypothesis in favor of the alternative that the sons are significantly taller than their fathers. The test statistic and the p-value can be obtained in `R` using: `t.test(height_son,height_father,paired=T,alternative="greater")`

Suppose we observe a random sample of $X_1, \dots, X_{n_1}$ and an independent sample of $Y_1, \dots, Y_{n_2}$. Such a situation can describe the value of a certain marker measured on two independent groups of patients, each undergoing a different treatment. We want to determine whether the marker is equal for both treatment groups, or whether it differs significantly. Suppose that all variables are normally distributed with $X_i \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_i \sim \mathrm{N}(\mu_2, \sigma_2^2)$. We want to test $H_0 : \mu_1 = \mu_2$. It can be shown that if H_0 holds, the statistic

$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_\bullet}$$

follows the Student's t-distribution. The sample standard deviation $s_\bullet$ and the number of degrees of freedom depend on whether the samples have equal variances ($\sigma_1^2 = \sigma_2^2$) or not. The test can then be performed by comparing the test statistic T with the corresponding critical values of the t-distribution.

Let $X_1, \dots, X_{n_1}$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_1, \dots, Y_{n_2}$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$.

Tests for the equality of expectations under $\sigma_1^2 = \sigma_2^2$:

H_0	H_A	test statistic T	critical region W_α
$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$		$\lvert T\rvert > t_{\alpha/2,n_1+n_2-2}$
$\mu_1 \leq \mu_2$	$\mu_1 > \mu_2$	$T = \dfrac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_{12}}\sqrt{\dfrac{n_1 n_2}{n_1 + n_2}}$	$T > t_{\alpha,n_1+n_2-2}$
$\mu_1 \geq \mu_2$	$\mu_1 < \mu_2$		$T < -t_{\alpha,n_1+n_2-2}$

- Where $s_{12} = \sqrt{\dfrac{(n_1-1)s_X^2 + (n_2-1)s_Y^2}{n_1+n_2-2}}$,
- t_{α,n_1+n_2-2} is the critical value of Student's t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Let $X_1, \ldots, X_{n_1}$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2}$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$.

Tests for the equality of expectations under $\sigma_1^2 \neq \sigma_2^2$:

H_0	H_A	test statistic T	critical region W_α
$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$	$T = \dfrac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_d}$	$\lvert T\rvert > t_{\alpha/2, n_d}$
$\mu_1 \leq \mu_2$	$\mu_1 > \mu_2$		$T > t_{\alpha, n_d}$
$\mu_1 \geq \mu_2$	$\mu_1 < \mu_2$		$T < -t_{\alpha, n_d}$

- Where $s_d = \sqrt{\dfrac{s_X^2}{n_1} + \dfrac{s_Y^2}{n_2}}$,
- $n_d = \dfrac{s_d^4}{\frac{1}{n_1-1}\left(\frac{s_X^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_Y^2}{n_2}\right)^2}$

Example 11.3 (– comparing men's heights from different countries)**.** Suppose we want to determine whether the average men's height is the same in the Czech Republic and in Norway. We have observed five men from CZE and six men from NOR. Their heights were measured as follows (in centimeters):

height CZE	X_i	169	178	179	186	191	
height NOR	Y_i	175	182	183	189	191	192

We test whether the expected heights are equal, against the alternative that they are not, on $\alpha = 5\%$. We take the variances as equal. The test statistic using equal variances is

$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2}}{s_{12}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}} = -1.0545.$$

The $\alpha/2$ critical value of the Student's t-distribution with n_1+n_2-2 degrees of freedom is $t_{\alpha/2, n_1+n_2-2} = t_{0.025,9} = 2.262$. Since

$$1.0545 = |T| < t_{\alpha/2, n_1+n_2-2} = 2.262,$$

we do not reject the null hypothesis of equality. Based on our data we could not find a significant difference between the expected heights of men among the two countries. The test statistic and the p-value can be obtained in `R` using: `t.test(height_cze,height_nor,paired=F,alternative="two.sided")`

Let $X_1, \ldots, X_{n_1}$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2}$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$.

Tests for the equality of variances – F-test:

H_0	H_A	test statistic T	critical region W_α
$\sigma_1^2 = \sigma_2^2$	$\sigma_1^2 \neq \sigma_2^2$	$T = \dfrac{s_X^2}{s_Y^2}$	$T < F_{1-\alpha/2, n_1-1, n_2-1} \vee T > F_{\alpha/2, n_1-1, n_2-1}$
$\sigma_1^2 \leq \sigma_2^2$	$\sigma_1^2 > \sigma_2^2$		$T > F_{\alpha, n_1-1, n_2-1}$
$\sigma_1^2 \geq \sigma_2^2$	$\sigma_1^2 < \sigma_2^2$		$T < F_{1-\alpha, n_1-1, n_2-1}$

- s_X^2 is the sample variance of the first random sample and s_Y^2 is the sample variance of the second sample.
- F_{α,n_1-1,n_2-1} is the critical value of the *Fisher-Snedecor F-distribution* with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.
- Important note: The F-test is particularly sensitive to the normality of X and Y. If we are not sure whether the data is normally distributed, it is better to use a different test or assume non-equal variances for the t-test.
- The test can be called in `R` using `var.test(height_cze,height_nor)`.