

Conditional probability and independence

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Probability and Statistics
BIE-PST, WS 2025/26, Lecture 2



Content

- **Probability theory:**

- ▶ Events, probability, **conditional probability, Bayes' Theorem, independence of events.**
- ▶ Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, covariance and correlation, important discrete and continuous distributions.
- ▶ Random vectors, joint and marginal distributions, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
- ▶ Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.

- **Mathematical statistics:**

- ▶ Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
- ▶ Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

A random experiment is represented using a probability space $(\Omega, \mathcal{F}, P)$:

- Ω is the set of possible results;
- $\mathcal{F}$ is a system of subsets of Ω ;
- elements $A \in \mathcal{F}$ are called random events;
- the probability measure P is a function, which assigns to the random events a real value from 0 to 1, representing the ideal proportion of cases, in which the events occur.

If there is only a finite many possible results with equal probabilities, then

$$P(A) = \frac{|A|}{|\Omega|}.$$

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If we know that an even number was rolled, then it is clear that $P(4 \mid \text{even}) = 1/3$.

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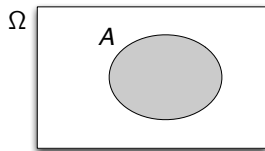
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Definition

Let A, B be events and $P(B) > 0$. The **conditional probability** of the event A given (the event) B is denoted by $P(A|B)$ and is defined as

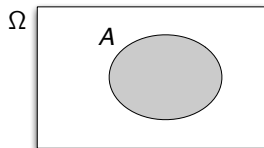
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Conditional probability – Venn diagram

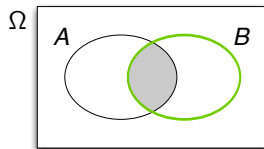


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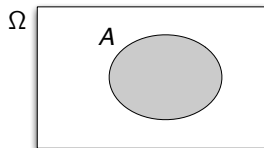


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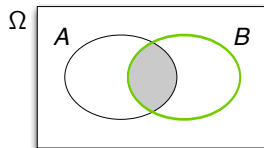


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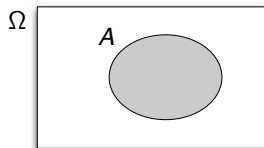
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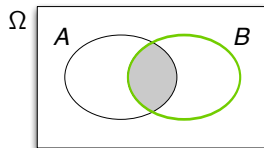
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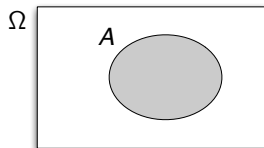
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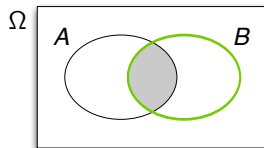
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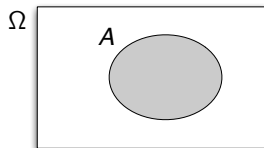


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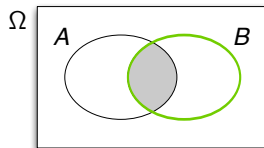
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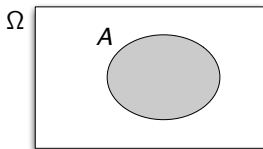
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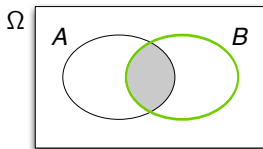
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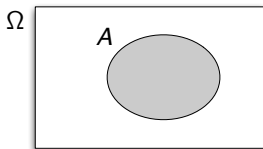
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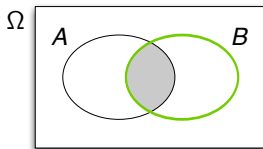
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$$P(\text{intersection}) = P(\text{event} \mid \text{condition}) P(\text{condition})$$

Conditional probability – examples

Example – rolling two dice

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Formally: $\Omega = \{1, 2, 3, 4, 5, 6\}^2$,

$P(A) = |A|/36$ for each $A \subset \Omega$.

Let $B = \{(3, \omega_2) : 1 \leq \omega_2 \leq 6\}$, $A = \{(\omega_1, \omega_2) : \omega_1 + \omega_2 > 6\}$.

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Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{|A \cap B|}{36}}{\frac{|B|}{36}} = \frac{|A \cap B|}{|B|} = \frac{3}{6}.$$

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Example – family with two children

A trickier example:

A family has two children. What is the probability that both are boys, given that at least one of them is a boy? I.e., what is the value of $P(\text{both boys} \mid \text{at least one is a boy})$?

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Incorrect: $P(BB \mid \text{older is boy}) = P(BB \mid BG \cup BB) = \frac{P(BB \cap (BG \cup BB))}{P(BG \cup BB)} = \frac{1}{2}.$

Properties of conditional probability

Lemma

Let $P(B) > 0$. Then the conditional probability $P(\cdot|B)$ is a probability measure, i.e., $P(\cdot|B) \in [0, 1]$ and it fulfills the axioms of probability.

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- iv) **σ -additivity**: If $A_1, A_2, \dots \in \mathcal{F}$ are mutually disjoint events (i.e., $A_i \cap A_j = \emptyset$ for $\forall i, j : i \neq j$), then

$$P\left(\bigcup_{i=1}^{+\infty} A_i \middle| B\right) = \frac{P\left(\left(\bigcup_{i=1}^{+\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{+\infty} (A_i \cap B)\right)}{P(B)} = \dots = \sum_{i=1}^{+\infty} P(A_i|B).$$



Properties of conditional probability

Conditional probability fulfills all mentioned properties of probability as well:

- if A_1 and A_2 are mutually disjoint, then $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$,
- $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$,
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Moreover, the probability $P(A|B)$ “lives” on B : for $A \cap B = \emptyset$ we have $P(A|B) = 0$.

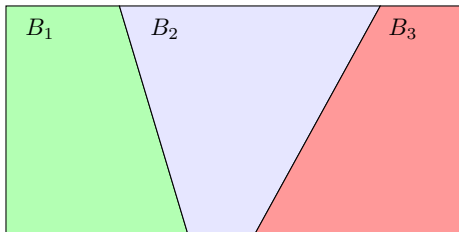
$$\text{Furthermore, } P(A \cap B|B) = \frac{P(A \cap B \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = P(A|B).$$

Case distinct formula (Law of total probability)

$$\Omega = B_1 \cup B_2 \cup B_3 \text{ (disjoint partition)}$$

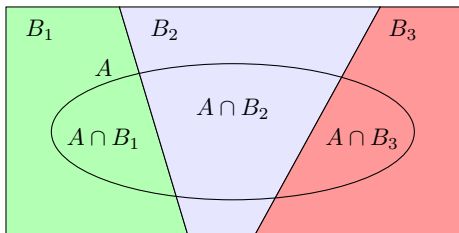
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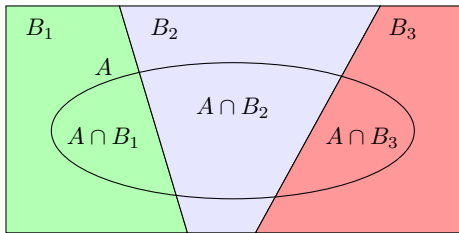
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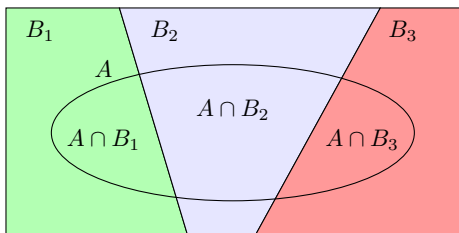


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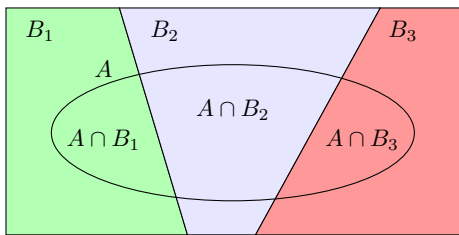
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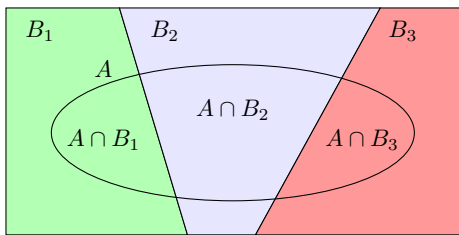
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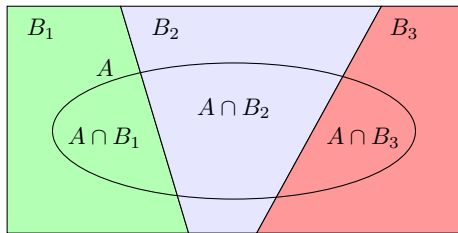
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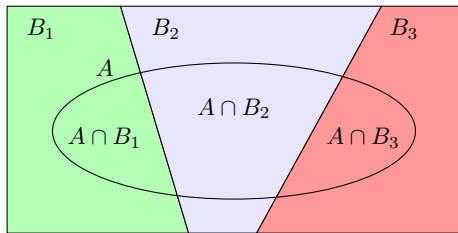
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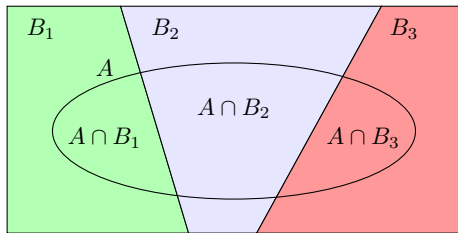
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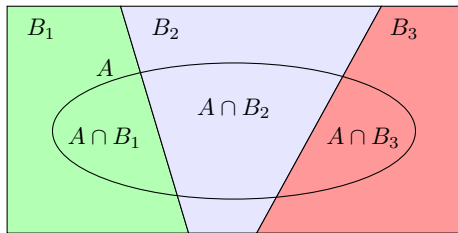
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Bayes' Theorem = converse procedure

At the end we observe A and we ask ourselves, what is the probability that the event B_j occurred.

$\Omega = B_1 \cup B_2 \cup B_3$ (disjoint partition)



Recall:

$$P(A \cap B_j) = P(A|B_j) P(B_j)$$

$$P(A) = P(A|B_1) P(B_1) + P(A|B_2) P(B_2) + P(A|B_3) P(B_3)$$

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Let $B_1, B_2, \dots, B_n$ be a partition of Ω such that $\forall i : P(B_i) > 0$.

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Bayes' Theorem – example

Example – spam filter

From the analysis of our email account we find out that:

- 30% of all delivered messages is spam;
- in 70% of spam messages there is the word “copy”;
- only in 10% of non-spam messages there is the word “copy”.

Calculate the probability that a message containing the word “copy” is a spam,

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S : set of spam messages,

$S^c = \Omega \setminus S$: set of non-spam messages,

C : set of messages containing word “copy”,

C^c : set of messages not containing the word “copy”.

$$P(S) = 0.3, P(S^c) = 0.7, \quad P(C|S) = 0.7, P(C|S^c) = 0.1$$

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$$P(S|C) = \frac{P(C|S) P(S)}{P(C|S) P(S) + P(C|S^c) P(S^c)} = \frac{0.7 \cdot 0.3}{0.7 \cdot 0.3 + 0.1 \cdot 0.7} = \frac{21}{28} = 0.75.$$

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which can be proven by using the definition of conditional probability on the right hand side:

$$\begin{aligned} P(A) P(B|A) P(C|A \cap B) &= P(A) \frac{P(B \cap A)}{P(A)} \frac{P(C \cap (A \cap B))}{P(A \cap B)} \\ &= P(A \cap B \cap C). \end{aligned}$$

Probability trees

Lemma – Multiplicative law

Let for events $A_1, \dots, A_n$ hold that $P(A_1 \cap \dots \cap A_n) > 0$. Then it holds that

$$P(A_1 \cap \dots \cap A_n) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap \dots \cap A_{n-1}).$$

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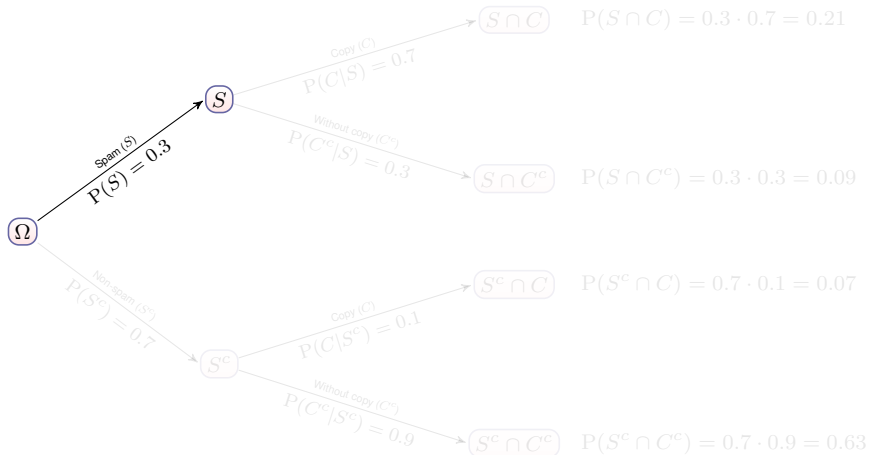
Proof

We apply successively the relation $P(A \cap B) = P(A) P(B|A)$ following from the definition of conditional probability:

$$\begin{aligned} P(A_1 \cap \dots \cap A_n) &= P(A_1 \cap \dots \cap A_{n-1}) P(A_n|A_1 \cap \dots \cap A_{n-1}) \\ &= P(A_1 \cap \dots \cap A_{n-2}) P(A_{n-1}|A_1 \cap \dots \cap A_{n-2}) P(A_n|A_1 \cap \dots \cap A_{n-1}) \\ &= \dots \end{aligned}$$

□

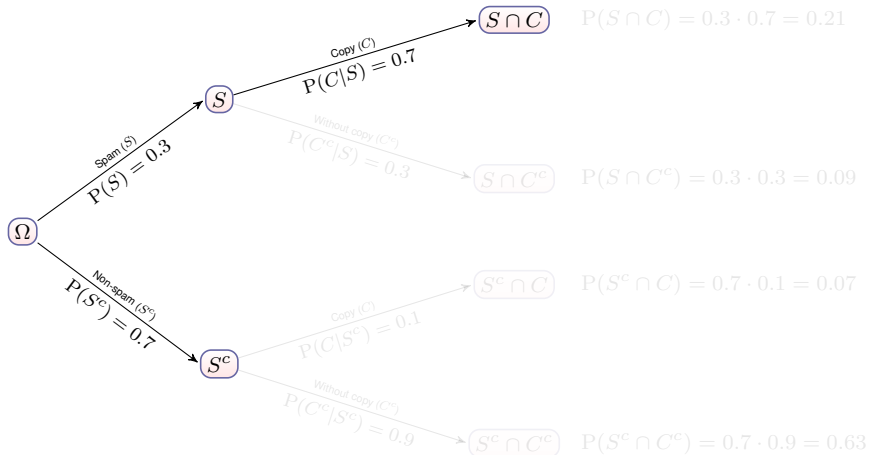
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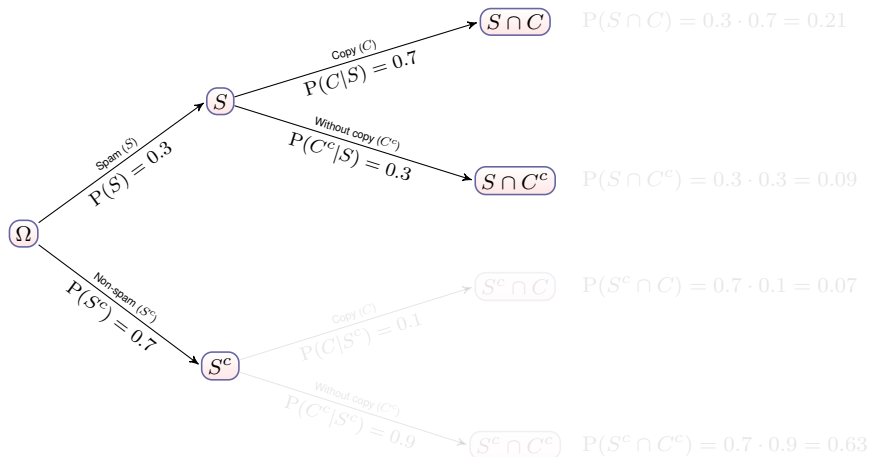
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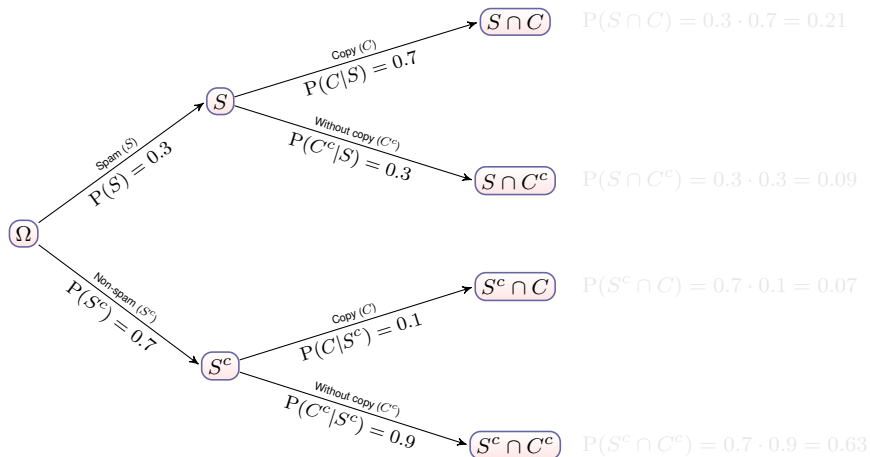
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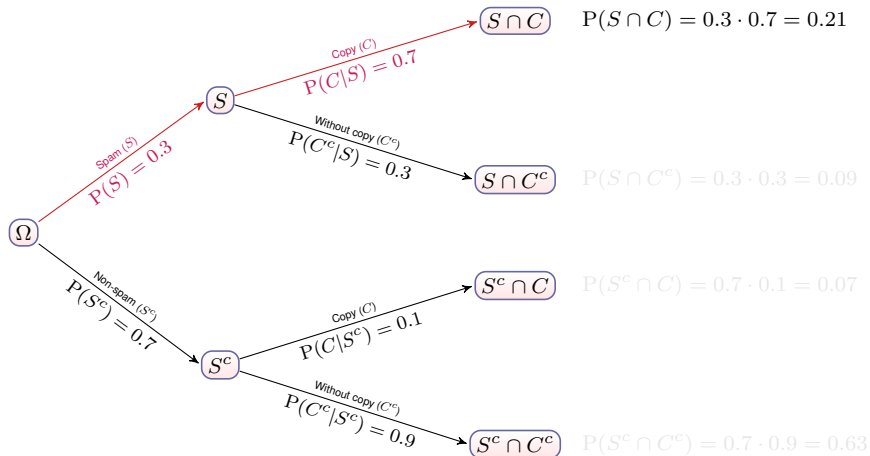
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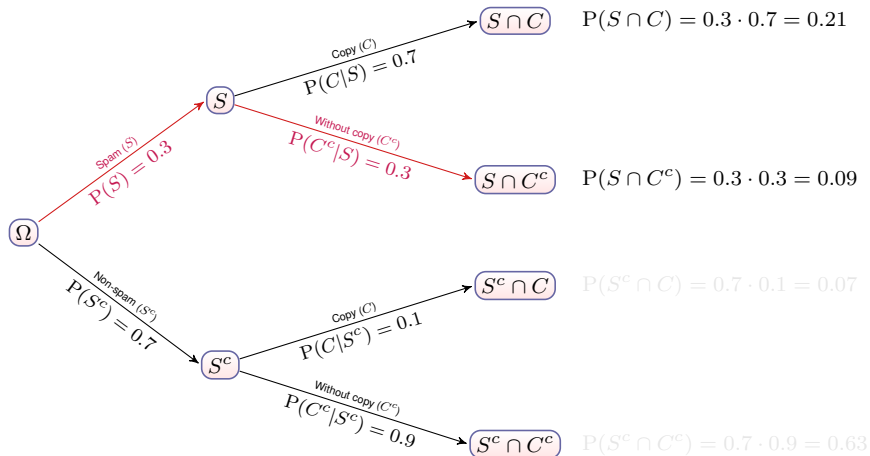
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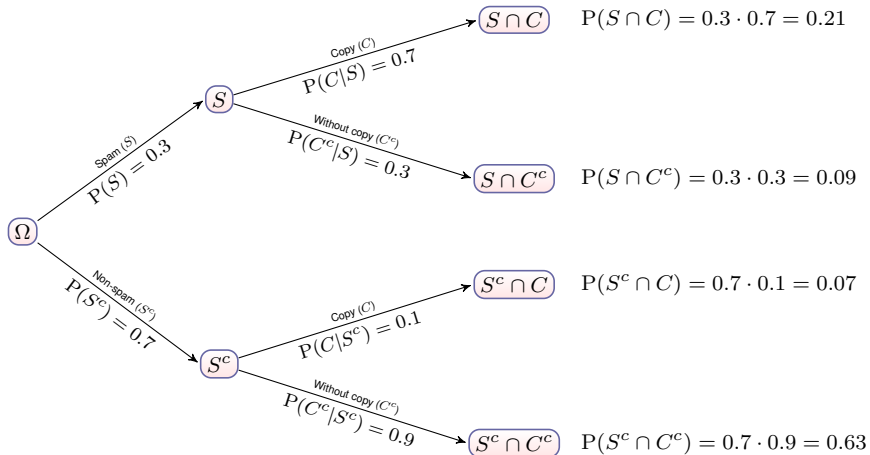
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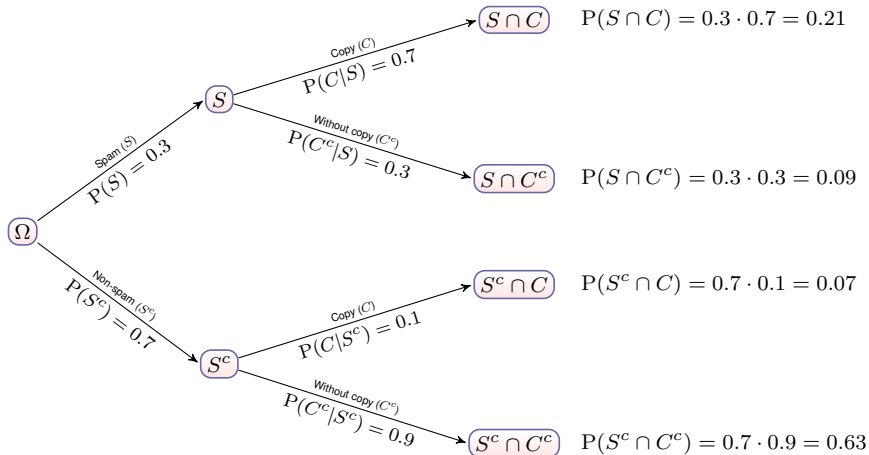
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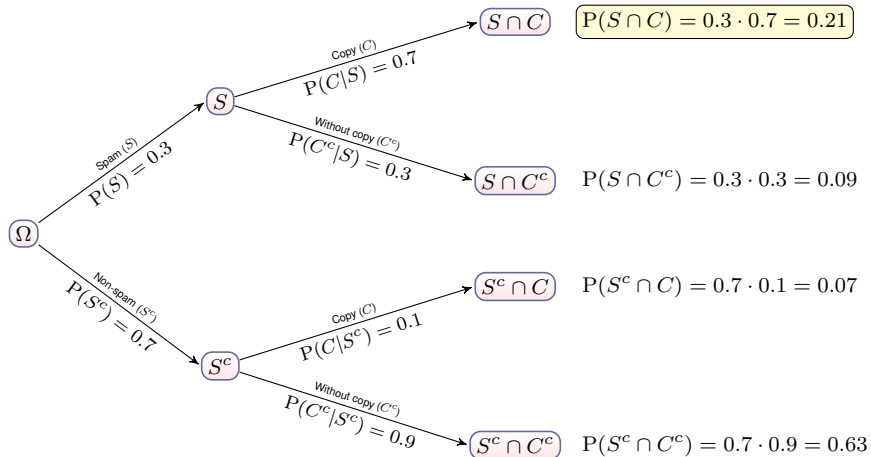


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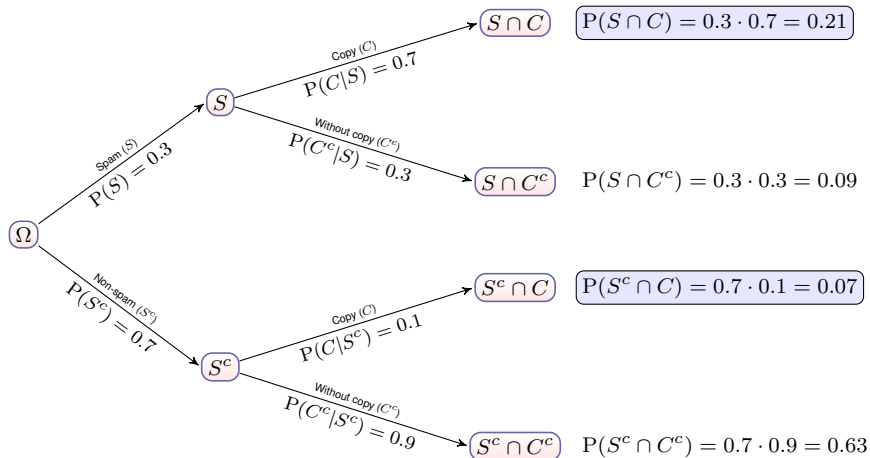
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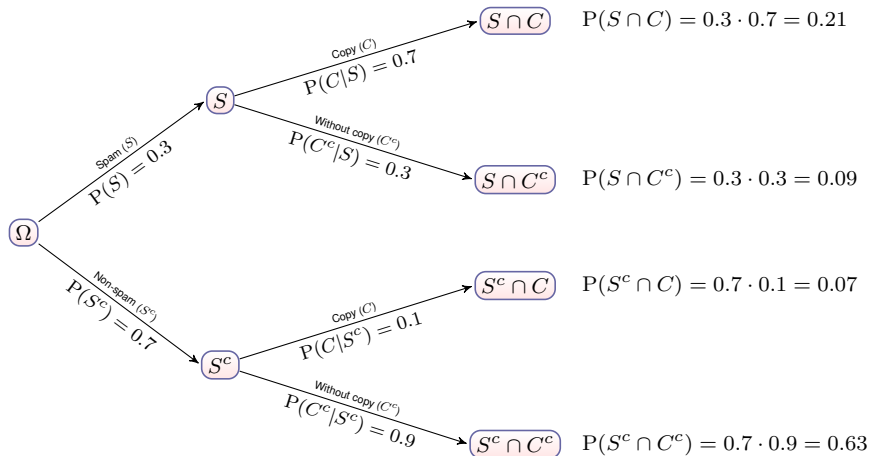
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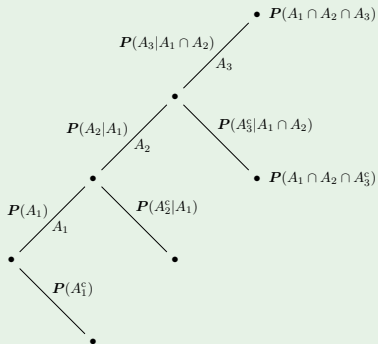
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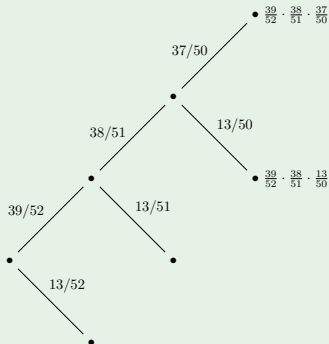
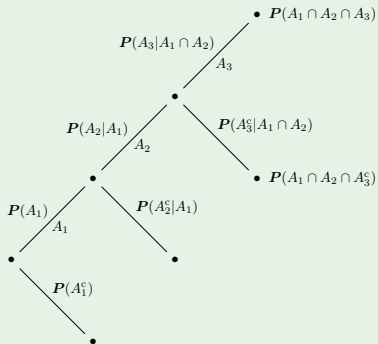


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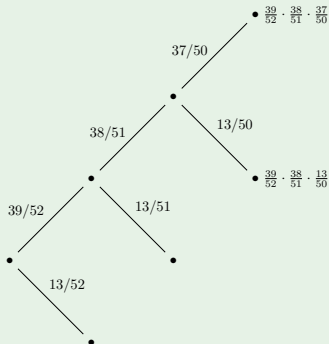
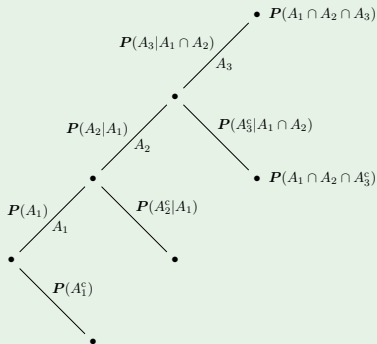


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The probability of a given vertex of the tree is the product of the corresponding values on the path stemming from the root.

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$$P(\text{drunk}|\text{accident}) = 0.1.$$

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From a [different study](#), we have found out that less than 1% of drivers are driving under influence. The overall chance of accident is difficult to determine, so we will compute just how more likely it is to cause an accident for drunk drivers:

$$\begin{aligned}\frac{P(\text{accident}|\text{drunk})}{P(\text{accident}|\text{sober})} &= \frac{P(\text{accident} \cap \text{drunk}) / P(\text{drunk})}{P(\text{accident} \cap \text{sober}) / P(\text{sober})} \\ &= \frac{P(\text{drunk}|\text{accident}) \cdot P(\text{accident}) / P(\text{drunk})}{P(\text{sober}|\text{accident}) \cdot P(\text{accident}) / P(\text{sober})} = \frac{0.1/0.01}{0.9/0.99} = 11.\end{aligned}$$

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Drunk drivers have at least 11 times higher probability of causing a fatal accident.

Independence of events

Intuitively: A and B are independent if the probability of the event A is not influenced by the knowledge about occurrence of the event B , i.e., $P(A|B) = P(A)$, and (vice versa) $P(B|A) = P(B)$.

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Generally, a family of events $\{A_i \mid i \in I\}$ is called **independent** if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

for all finite non-empty subsets J of I .

Independence of events

Example – rolling a die

Consider the events

A : "an even number is rolled" and B : "a number less than 3 is rolled".

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$$P(A \cap B) = \frac{1}{6} \quad \text{and} \quad P(A)P(B) = \frac{3}{6} \cdot \frac{1}{6} = \frac{1}{12}.$$

Then events A and B are not independent.

Relation between independence and conditional probability

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For A and B independent the knowledge of B does not bring us any information about A .

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If the events A and B are independent then A and B^c (resp., A^c and B ; A^c and B^c) are independent, too.

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Theorem

*If $(A_i)_{i \in I}$ is a family of independent events, then **for any arbitrary non-empty finite subset $\emptyset \neq J \subset I$** it holds that*

$$P \left(\bigcap_{i \in J} A_i \mid \bigcap_{i \in I \setminus J} A_i \right) = P \left(\bigcap_{i \in J} A_i \right).$$

Independent vs disjoint events

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The events being disjoint is a matter of sets, independence is a matter of probabilities.

Conditional independence

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and C an event with $P(C) > 0$. Events A and B are called **conditionally independent with respect to C** , if

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Recall:

- $Q(A) = P(A | C)$ is a probability measure;
- the conditional independence is thus the independence with respect to probability Q .

Conditional independence

Example – rolling a seven-sided die

Suppose we roll a seven-sided die with all sides equally likely. Consider the events:

A : "an even number is rolled", B : "a number less than 3 is rolled".

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Example – rolling a seven-sided die + condition

Consider further event C : "we rolled at most 6" $C = \{1, 2, 3, 4, 5, 6\}$.

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Events A and B are not independent.

Example – rolling a seven-sided die + condition

Consider further event C : "we rolled at most 6" $C = \{1, 2, 3, 4, 5, 6\}$.

Are events A and B conditionally independent with respect to C ?

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(\{2\})}{P(\{1, \dots, 6\})} = \frac{1/7}{6/7} = \frac{1}{6},$$

$$P(A|C) \cdot P(B|C) = \frac{3/7}{6/7} \cdot \frac{2/7}{6/7} = \frac{1}{6}.$$

Events A and B are conditionally independent with respect to C .

Recap

- The **conditional probability** that an event A occurs if we know that an event B with $P(B) > 0$ occurred, is defined as $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

- **Law of total probability**: For A and B with $P(B) > 0$ we have

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

- **Bayes' Theorem**: For A and B with $P(A) > 0$ and $P(B) > 0$ we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Recap

- The **conditional probability** that an event A occurs if we know that an event B with $P(B) > 0$ occurred, is defined as $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

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- Events A and B are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

- For **independent** events A and B the knowledge that one of them occurred or not does not change the probability of the other one happening:

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B).$$