

Random variables I.

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Probability and Statistics

BIE-PST, WS 2025/26, Lecture 3



Content

- **Probability theory:**

- ▶ Events, probability, conditional probability, Bayes' Theorem, independence of events.
- ▶ **Random variables, distribution function, functions of random variables**, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, covariance and correlation, important discrete and continuous distributions.
- ▶ Random vectors, joint and marginal distributions, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
- ▶ Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.

- **Mathematical statistics:**

- ▶ Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
- ▶ Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

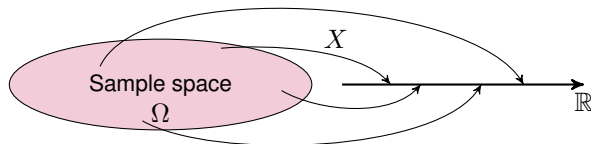
A random experiment is represented using a probability space $(\Omega, \mathcal{F}, P)$:

- Ω is the set of possible outcomes ω .
- $\mathcal{F}$ is a system of subsets of Ω with desirable properties.
- Elements $A \in \mathcal{F}$ are called random events.
- Probability measure P is a function, which assigns real values from 0 to 1 to the random events. It represents the ideal proportion of cases, in which the events occur.

Random variable

For a mathematical processing of a random experiment it is often useful to assign a number to each outcome ω . By this assignment we choose the part of information which is interesting from our point of view.

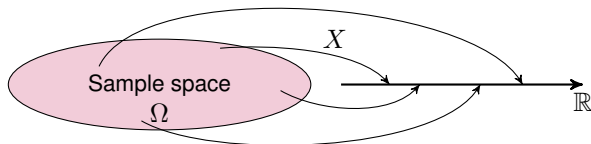
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Examples

- Number of Heads while tossing a coin: $X(\text{Heads}) = 1, X(\text{Tails}) = 0$.
- Number of winnings in the game with: $X(\text{Heads}) = 1, X(\text{Tails}) = -1$.
- How much a player won in a given game at a poker tournament.
- The highest rolled value or n rolls of a die.
- The height of a randomly chosen person.

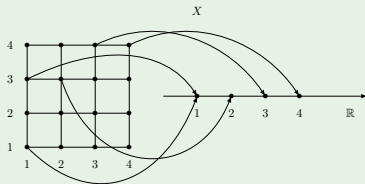
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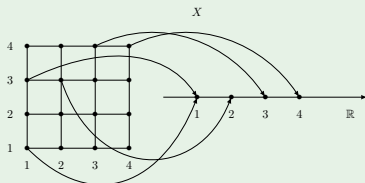
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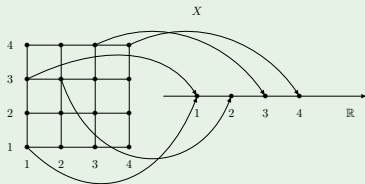
$$P(X = 1) = P(\{\omega | X(\omega) = 1\})$$

$$= P(\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 1), (4, 1)\}) = \frac{7}{16}.$$

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Similarly,

$$P(X = 2) = P(\{(2, 2), (2, 3), (2, 4), (3, 2), (4, 2)\}) = \frac{5}{16},$$

$$P(X = 3) = P(\{(3, 3), (3, 4), (4, 3)\}) = \frac{3}{16},$$

$$P(X = 4) = P(\{(4, 4)\}) = \frac{1}{16}.$$

Random variable and its distribution function

Definition

A **random variable** X on a probability space $(\Omega, \mathcal{F}, P)$ is a function $X : \Omega \rightarrow \mathbb{R}$, assigning to each outcome $\omega \in \Omega$ a number $X(\omega)$, with the property that:

$$\{X \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R}.$$

Such a function is said to be **$\mathcal{F}$ -measurable**.

- By the notation $\{X \leq x\}$ we mean the set $\{\omega \in \Omega : X(\omega) \leq x\}$.
- The measurability property in fact tells us that $\{X \leq x\}$ is **an event** and allows us to compute $P(X \leq x)$, $P(X = x)$, $P(X \in (a, b))$, etc.

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The **probability distribution** of a random variable is given by its distribution function:

Definition

The **distribution function** of a random variable X is a function $F: \mathbb{R} \rightarrow [0, 1]$ defined as

$$F(x) = P(X \leq x).$$

Random variable and its distribution function

There are various types of random variables.

- Some can take only isolated values (e.g., 0 or 1 for Heads and Tails of a coin toss, $1, \dots, 6$ for a die roll).

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Regardless of the type, the distribution function gives us a full description of the random variable.

For any real number x , we can answer the question: "what is the probability that the random variable will be less than or equal to x "?

This allows us to answer questions about any equalities and inequalities.

Properties of the distribution function

Theorem

The distribution function F of a random variable X has following properties:

- i) F is non-decreasing: *if $x < y$, then $F(x) \leq F(y)$*
- ii) F “starts at 0 and ends at 1”: $\lim_{x \rightarrow -\infty} F(x) = 0$ *and* $\lim_{x \rightarrow \infty} F(x) = 1$
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Proof

- i) Recall the notation $\{X \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\}$. Consider the disjoint partition

$$\{X \leq y\} = \{X \leq x\} \cup \{x < X \leq y\},$$

therefore $F(y) = P(X \leq y) = P(X \leq x) + P(x < X \leq y) \geq P(X \leq x) = F(x)$.

- ii) For simplicity we only sketch the proof by means of a sequence of events $B_n = \{X \leq -n\}$. For $n \rightarrow \infty$ it is decreasing in the sense of inclusion with the intersection equal to $\emptyset$, i.e., $B_n \searrow \emptyset$. From the continuity of probability theorem we have $P(B_n) \rightarrow P(\emptyset) = 0$. For the proof of the second statement it is enough to consider a sequence $A_n = \{X \leq n\} \nearrow \Omega$ and from the same theorem we have $P(A_n) \rightarrow P(\Omega) = 1$.

- iii) Similarly as ii) (see bibliography).



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By means of the distribution function it is possible to express some important properties.

Lemma

Let F be a distribution function of a random variable X , then it holds that:

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Proof

- i) $\Omega = \{X > x\} \cup \{X \leq x\}$ is a disjoint partition. Therefore $P(\{X > x\}) = P(\{X \leq x\}^c).$
- ii) See proof of i) of the previous theorem.
- iii) See bibliography. Idea of the proof using a non-decreasing sequence and continuity of probability:

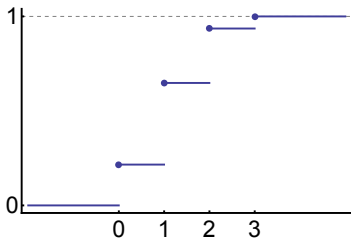
$$\{X \leq x - 1/n\} \nearrow \{X < x\} \quad \Rightarrow \quad F(x - 1/n) = P(X \leq x - 1/n) \rightarrow P(X < x).$$
- iv) $\{X \leq x\} = \{X < x\} \cup \{X = x\}$ is a disjoint partition. Therefore

$$P(X = x) = P(X \leq x) - P(X < x).$$

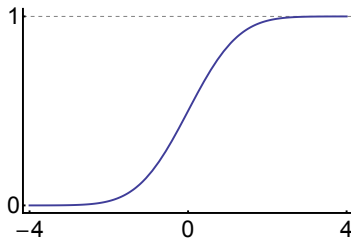


Types of random variables and their distribution functions

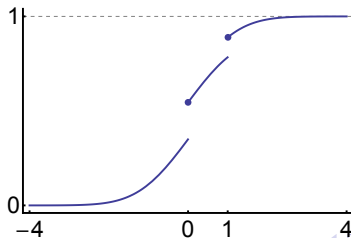
Discrete random variable



Continuous random variable



Mixed random variable



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Probabilities of the values of a discrete random variable X are given by

$$P(X = x_k), \quad k = 1, 2, \dots$$

The probabilities $P(X = x_k)$ can be viewed as a function of x and are sometimes called a **probability function**, or a **probability mass function** or a **discrete density** of the variable X .

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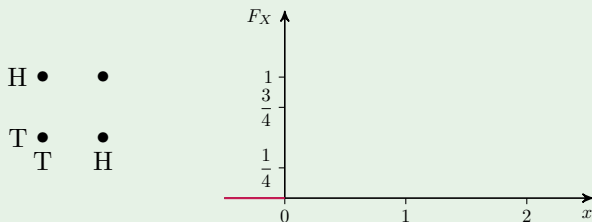
$$F_X(x) = P(X \leq x) = \sum_{\text{all } x_k \leq x} P(X = x_k).$$

From this it follows that $F_X(x)$ has jumps at points x_k and it is constant elsewhere. The size of the jump at point x_k is equal to $P(X = x_k)$.

Example of a discrete random variable

Example – toss with two coins

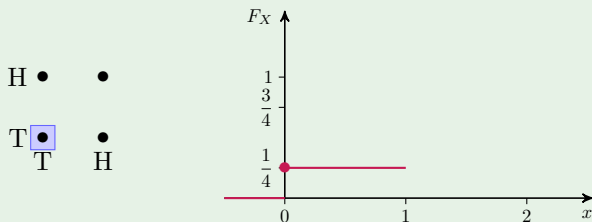
The sample space is $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. Let the random variable X give the number of Heads. The distribution function is $F_X(x) = P(X \leq x)$:



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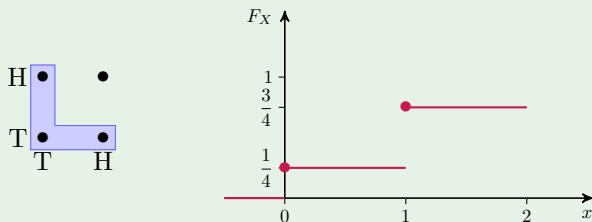
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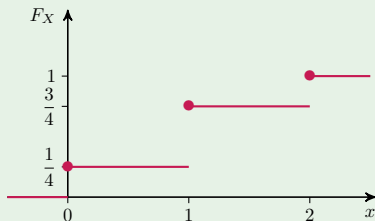
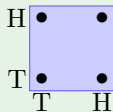
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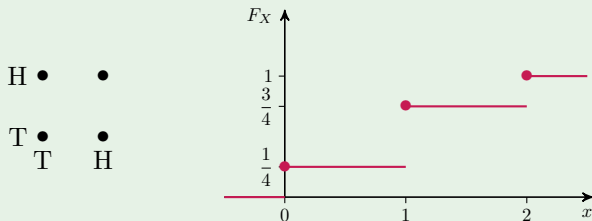
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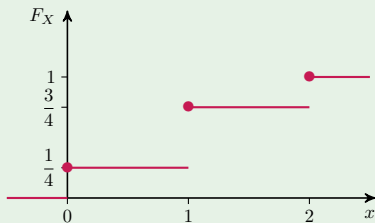
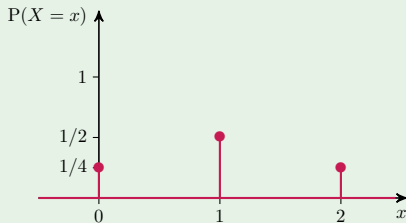
$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 & P(\emptyset) \\ 1/4 & \text{for } 0 \leq x < 1 & P(\{(T, T)\}) \\ 3/4 & \text{for } 1 \leq x < 2 & P(\{(T, T), (H, T), (T, H)\}) \\ 1 & \text{for } 2 \leq x & P(\Omega). \end{cases}$$

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The sample space is $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. Let the random variable X give the number of Heads.

Draw the probabilities of the values and the distribution function.

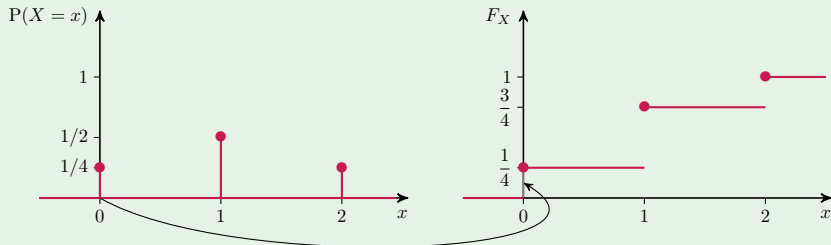


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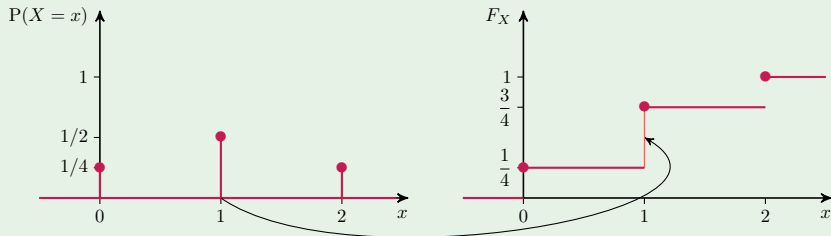


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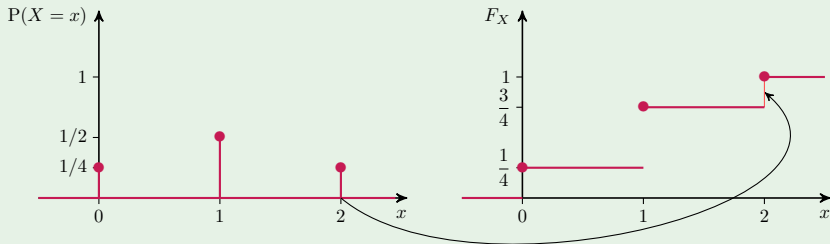


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Computation of the probabilities $P(X = x_k)$:

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Computation of the distribution function $F_X(x) = P(X \leq x_k)$:

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Examples of discrete random variables

Remark

A random variable X can be discrete even if the sample space itself is not discrete.

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Example

Let us throw darts at a target $T \subset \mathbb{R}^2$.

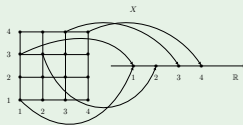
The target can be divided into parts (often concentric annulus), denoted as T_1, T_2, T_3, T_4, T_5 .

We can consider a discrete random variable X denoting the points obtained from one throw, for example

$$X(\omega) = \begin{cases} 10 & \text{for } \omega \in T_5 \\ 5 & \text{for } \omega \in T_4 \\ i & \text{for } \omega \in T_i, i = 1, 2, 3 \end{cases}$$

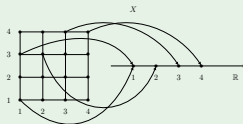
Example – minimum of two rolls of a 4-sided die (continuation)

$X = \min\{1^{\text{st}} \text{ roll}, 2^{\text{nd}} \text{ roll}\}:$

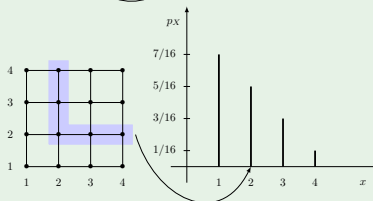


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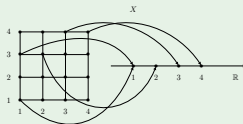


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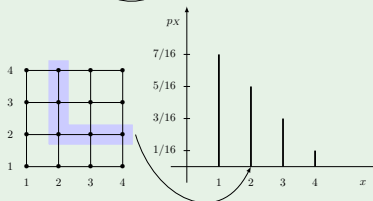


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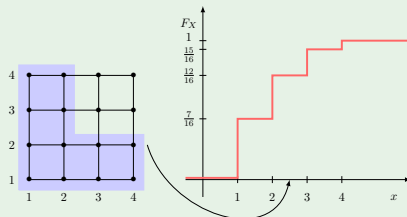
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Probabilities:



Distribution function:



Important discrete probability distributions

Example

(Will be studied later)

- **Bernoulli** (Alternating) distribution with a parameter $p \in [0, 1]$, $X \sim \text{Be}(p)$:
(One toss of an unbalanced coin.)

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

- **Binomial** distribution with parameter $p \in [0, 1]$, $X \sim \text{Binom}(n, p)$:
(Number of Heads in n tosses of an unbalanced coin.)

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

- **Geometric** distribution with a parameter $p \in (0, 1)$, $X \sim \text{Geom}(p)$:
(Number of tosses of an unbalanced coin until the first Heads appear.)

$$P(X = k) = (1 - p)^{k-1} p$$

- **Poisson** distribution with a parameter $\lambda > 0$, $X \sim \text{Poisson}(\lambda)$:
(Limit of the Binomial distribution for $n \rightarrow \infty$.)

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Continuous random variables – motivation

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Therefore we regard each singular value as having **zero probability** (intuitively, it is, e.g., infinitely improbable having to wait for the bus for exactly 3 : 00 : 00... minutes).

Instead, we need a way to measure the probability of **intervals**.

Recall the Romeo and Juliet problem, where each of them arrives at a random time point in an one-hour window, evenly chosen.

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Often we need to introduce an uneven distribution of values.

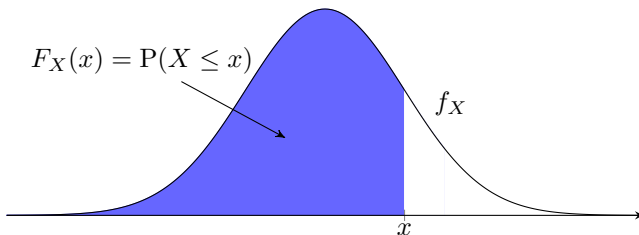
Continuous random variables

Definition

A random variable X is called (absolutely) **continuous**, if there exists a **non-negative** function $f_X : \mathbb{R} \rightarrow [0, +\infty)$ such that for all $x \in \mathbb{R}$ the distribution function F_X can be expressed as

$$F_X(x) = \int_{-\infty}^x f_X(t) dt.$$

The function f_X is called the **probability density** of the random variable X .



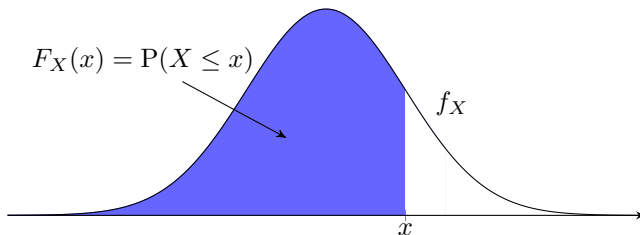
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The **distribution function** of a continuous random variable is **continuous**.

Properties of continuous random variables

Theorem

Let f_X be a density of a continuous random variable X . Then it holds that

- i) $\int_{-\infty}^{+\infty} f_X(t) dt = 1$ (**normalization condition**),
- ii) $P(X = x) = 0$ for all $x \in \mathbb{R}$,
- iii) $f_X(t) = \frac{dF_X}{dt}(t)$ at points where the derivative exists,
- iv) $P(a < X \leq b) = \int_a^b f_X(t) dt = F_X(b) - F_X(a)$,
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Consequences:

- $P(X \leq x) = P(X < x)$ – from ii)
- $f_X(t) dt \approx P(t < X < t + dt)$ for $dt \ll 1$ – from iv)

Properties of continuous random variables

Proof

i)
$$\int_{-\infty}^{+\infty} f_X(x) dx = \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

ii) Using the continuity of the distribution function and the previous theorem:

$$P(X = x) = F(x) - \lim_{y \rightarrow x-} F(y) = 0.$$

iii) It follows from the properties of derivatives and integrals (first fundamental Theorem of calculus).

iv)
$$P(a < X \leq b) = F(b) - F(a) = \int_{-\infty}^b f_X(t) dt - \int_{-\infty}^a f_X(t) dt = \int_a^b f_X(t) dt.$$

(second fundamental Theorem of calculus – Newton's formula)

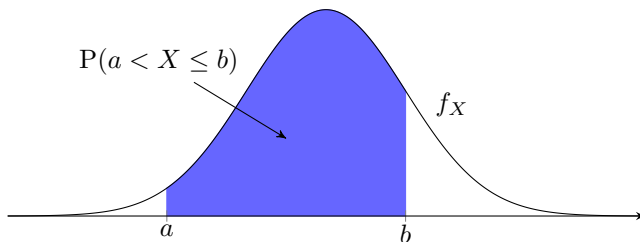
v) From the properties of the Lebesgue integral – advanced, see bibliography.



Relation between density and probability

Now we recall and illustrate the important property of the probability density:

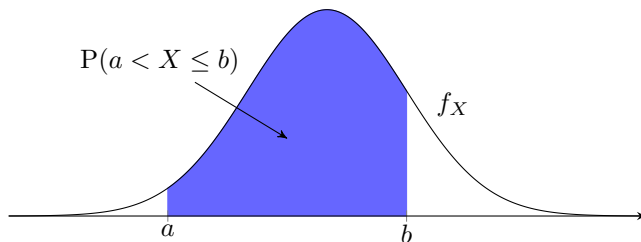
$$P(a < X \leq b) = \int_a^b f_X(x) \, dx = \left[F(x) \right]_a^b = F(b) - F(a).$$



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Note that when dealing with **continuous** random variables, it does not matter whether the inequalities are strict or non-strict.

$$P(a < X \leq b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b).$$

Romeo, Juliet and the uniform distribution

Example – uniform distribution of Romeo's arrival

Denote the time when Romeo arrives at the meeting point as a random variable X . Suppose that X has the **uniform distribution** on the interval $[0, 1]$, meaning that its density is constant on this interval and zero elsewhere.

$$f_X(x) = \begin{cases} c & \text{for } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

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From the normalization condition we know that the area under the graph of the density needs to be equal to one. Therefore the density needs to integrate to one:

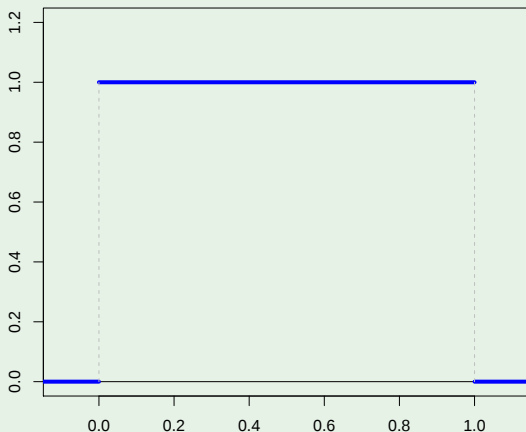
$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 c \cdot dx = [c \cdot x]_0^1 = c \cdot 1 - c \cdot 0 = c = 1.$$

The constant c has to be equal to one.

Romeo, Juliet and the uniform distribution

Example – uniform distribution of Romeo's arrival (continued)

Density of the continuous uniform distribution on the interval $[0, 1]$:



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What is the probability that Romeo arrives between 12:15 and 12:45?

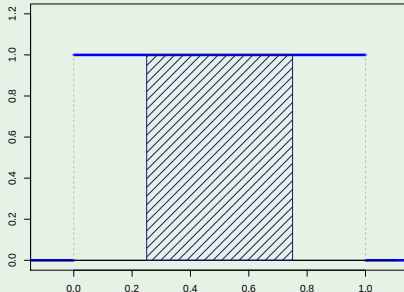
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Example – uniform distribution of Romeo's arrival (continued)

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Probabilities concerning intervals are obtained as the corresponding area under the density:

$$\int_{1/4}^{3/4} 1 dx = [x]_{1/4}^{3/4} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

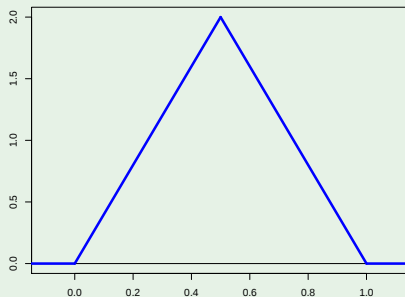


Romeo, Juliet and a non-uniform distribution

Example – non-uniform distribution of Juliet's arrival

Suppose that Juliet arrives at the meeting point according to a non-uniform distribution with density:

$$f_X(x) = \begin{cases} 4x & \text{for } x \in [0, 1/2] \\ 4 - 4x & \text{for } x \in [1/2, 1] \\ 0 & \text{otherwise.} \end{cases}$$



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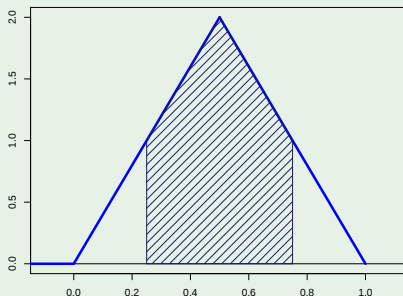
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Example – non-uniform distribution of Juliet's arrival (continued)

What is the probability that Juliet arrives between 12:15 and 12:45?

Probabilities concerning intervals are obtained as the corresponding area under the density:

$$\int_{1/4}^{3/4} f(x)dx = \dots = \frac{3}{4}.$$



Note that when the distribution of the arrivals is not uniform, the probability that they will meet cannot be obtained using the geometric approach as before.

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For a random variable X with a **known distribution**, we are often interested in the **distribution** of values somehow **calculated** from the values of X , say $Y = g(X)$.

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- The distribution of a random variable $g(X)$ can be discrete, continuous or mixed.

Function of a discrete random variable

Lemma – function of discrete random variable

Consider a function $g: \mathbb{R} \rightarrow \mathbb{R}$ and a **discrete** random variable X , and define the function of the random variable $g(X)$ by $g(X)(\omega) = g(X(\omega))$ for all $\omega \in \Omega$.

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Proof

The probabilities of the values of $g(X)$ can be obtained from the (countable) disjoint partition

$$\{g(X) = y\} = \bigcup_{x_k: g(x_k)=y} \{X = x_k\}.$$



Functions of random variables

Lemma – function of a general random variable

Consider a **measurable** function $g: \mathbb{R} \rightarrow \mathbb{R}$ and an **arbitrary** random variable X and define the function of random the variable $g(X)$ as $g(X)(\omega) = g(X(\omega))$ for all $\omega \in \Omega$.

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Proof

The proof that $g(X)$ is a random variable consists in verifying the measurability of $Y = g(X)$, i.e., that $\{Y \leq y\}$ is an event for all y :

$$\{g(X) \leq y\} = \{\omega \in \Omega: g(X(\omega)) \leq y\} \in \mathcal{F}, \forall y \in \mathbb{R}.$$

A detailed proof can be found in the bibliography. □

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Remark

Generally for a **distribution** function $F_Y(y)$ of a random variable $Y = g(X)$ it holds that

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- If g is strictly monotone, then g^{-1} is differentiable and $Y = g(X)$ is continuous with

$$f_Y(y) = f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}.$$

✓ **Proofs and more information can be found in bibliography.**

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- The generalized inverse of the distribution function is called the **quantile function** and can be used for simulations.