

Random vectors I.

(Random vectors, independence, conditional distribution)

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Probability and Statistics

BIE-PST, WS 2025/26, Lecture 6



Content

- **Probability theory:**

- ▶ Events, probability, conditional probability, Bayes' Theorem, independence of events.
- ▶ Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, important discrete and continuous distributions.
- ▶ Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
- ▶ Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.

- **Mathematical statistics:**

- ▶ Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
- ▶ Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

- A **random variable** X is a measurable function, which assigns real values to the outcomes of a random experiment.
- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

$$F_X(x) = P(X \leq x).$$

- There are two major types of random variables:
 - ▶ **Discrete**, taking only countably many possible values.
 - ▶ **Continuous**, taking uncountably many values from an interval.
- The distribution can be given by:
 - ▶ for discrete distributions by the **probabilities** of possible values $P(X = x_k)$.
 - ▶ for continuous distributions by the **density** f_X for which

$$F_X(x) = \int_{-\infty}^x f_X(t) dt.$$

Random vectors

Sometimes we can measure **several** random **variables** at once from **one result** of an experiment.

The individual variables can have different distributions and the values of the variables can be strongly mutually interconnected. It is appropriate to **describe** their **distribution together** as the so called **joint distribution**.

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Definition

Consider two random variables X and Y defined on the same probability space $(\Omega, \mathcal{F}, P)$. We define their **joint distribution function** $F_{X,Y}(x, y)$ as

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For n random variables $X_1, X_2, \dots, X_n \stackrel{\text{denote}}{=} \mathbf{X}$ we define the joint distribution function as

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$$F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \leq x_1 \cap \dots \cap X_n \leq x_n).$$

The couple (X, Y) or, n-tuple $(X_1, \dots, X_n)$, is called a **random vector**.

Example – joint distribution

Example

Let X and Y be random variables with a joint discrete distribution given by the following probabilities:

		x		
$P(X = x \cap Y = y)$		0.5	1	2
y	2	0.3	0.06	0.04
	1	0.4	0.15	0.05

Compute the joint distribution function $F_{X,Y}(x, y) = P(X \leq x \cap Y \leq y)$:

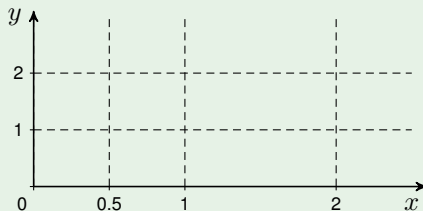
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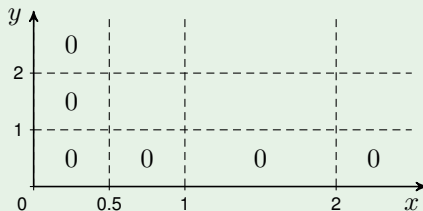
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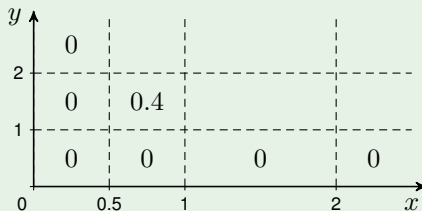
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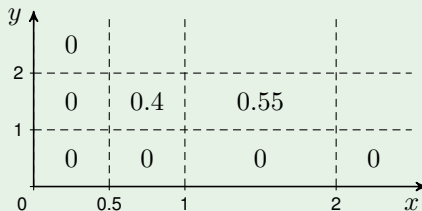
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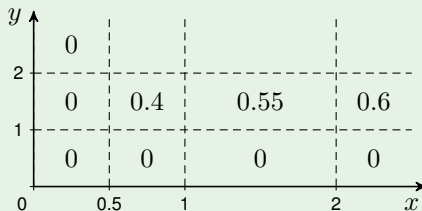
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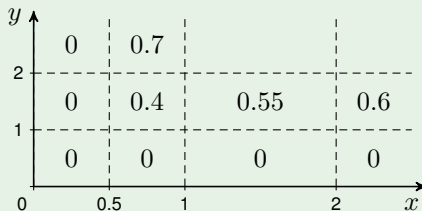
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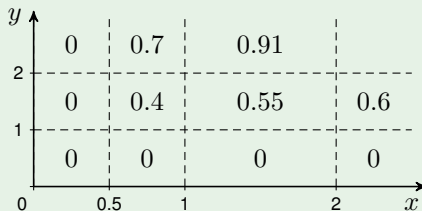
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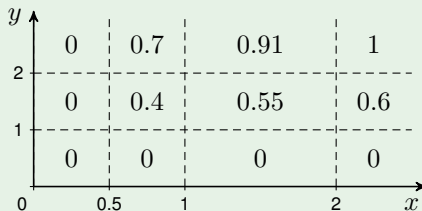
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Properties of the joint distribution function

The joint distribution function has analogous properties as the distribution function of one variable.

Theorem

The joint distribution function $F_{X,Y}$ of random variables X and Y has following properties:

- i) if $x_1 < x_2$ and $y_1 < y_2$ then $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$.*

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- iii) $\forall y \in \mathbb{R}, \lim_{x \rightarrow +\infty} F_{X,Y}(x, y) = F_Y(y)$ and
 $\forall x \in \mathbb{R}, \lim_{y \rightarrow +\infty} F_{X,Y}(x, y) = F_X(x)$.

Proof

Analogously as for the distribution function of one random variable. □

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If the variables X and Y are **discrete**, it is often useful to describe the distribution by the **joint probabilities** of their values.

Definition

The **joint probabilities of values** of two discrete random variables X and Y is

$$P(X = x \cap Y = y) = P(\{X = x\} \cap \{Y = y\}).$$

Taken as a function of x and y , the probabilities are called the **joint probability mass function**.

Joint probabilities and the joint distribution function

The **joint distribution function** of two discrete random variables X and Y is

$$F_{X,Y}(x,y) = P(X \leq x \cap Y \leq y) = \sum_{\{i: x_i \leq x\}} \sum_{\{j: y_j \leq y\}} P(X = x_i \cap Y = y_j)$$

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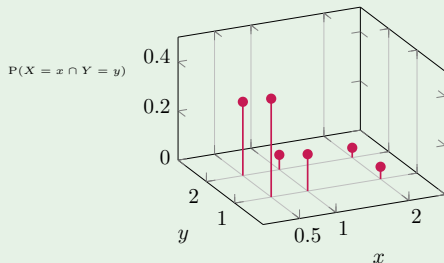
The **normalization condition** follows from the properties of the joint distribution function:

$$\begin{aligned} \sum_i \sum_j P(X = x_i \cap Y = y_j) &= \sum_i P\left(\{X = x_i\} \cap \bigcup_j \{Y = y_j\}\right) \\ &= \sum_i P\left(\{X = x_i\} \cap \{Y \in \mathbb{R}\}\right) = \sum_i P(X = x_i) \\ &= P\left(\bigcup_j \{X = x_i\}\right) = P(\{X \in \mathbb{R}\}) = P(\Omega) = 1. \end{aligned}$$

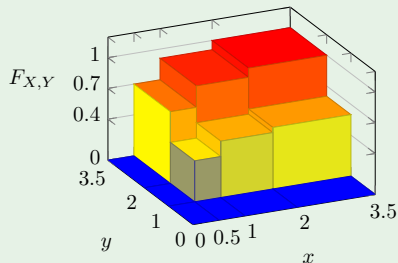
Joint distribution – visualization

Example

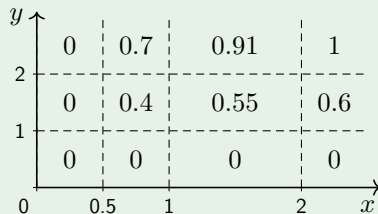
Joint probabilities



Joint distribution function



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Marginal distribution

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Let $P(X = x \cap Y = y)$ be the joint probabilities of values of two discrete variables X and Y . The **marginal distribution** (or **marginal probabilities**) of a X is given by

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$$\begin{aligned} P(X = x) &= P(\{X = x\} \cap \{Y \in \mathbb{R}\}) = P(\{X = x\} \cap (\bigcup_j \{Y = y_j\})) = \\ &= P(\bigcup_j (\{X = x\} \cap \{Y = y_j\})) = \sum_j P(\{X = x\} \cap \{Y = y_j\}). \end{aligned}$$



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Random variables forming a countable collection $X_1, X_2, \dots$ are called independent if all finite n -tuples $X_{i_1}, \dots, X_{i_n}$ are independent.

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Proof

If the condition regarding equalities holds, it must hold also for all inequalities, because they can be rewritten as sums of probabilities of disjoint events.

If the condition regarding inequalities hold, it must hold also for all equalities, because the difference of probabilities of inequalities yields probabilities of equalities. □

Example – checking independence of random variables

Example – continuation

Random variables X and Y have the following joint and marginal distributions:

$P(X = x \cap Y = y)$		x			$P(Y = y)$
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Are X and Y independent?

No, they are not independent because, e.g., for $x = 0.5$ and $y = 2$ it holds that

$$0.3 = P(X = 0.5 \cap Y = 2) \neq P(X = 0.5) \cdot P(Y = 2) = 0.7 \cdot 0.4 = 0.28.$$

Vectors of continuous random variables

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If the variables X and Y are **continuous**, it is often useful to describe the distribution by the **joint probability density**.

Definition

Two random variables X and Y have a **joint (absolutely) continuous** distribution if there exists a **non-negative** function $f_{X,Y} : \mathbb{R}^2 \rightarrow [0, +\infty)$ such that for all $x, y \in \mathbb{R}$ it holds

$$F_{X,Y}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(u, v) \, du \, dv.$$

The function $f_{X,Y}$ is called the **joint probability density** of the random variables X, Y or of the random vector (X, Y) .

Properties of continuous random variables

Similarly as in the one-dimensional case it holds that:

- Where the derivative exists:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y).$$

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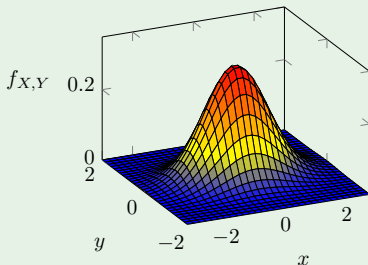
- $P(\{a < X \leq b\} \cap \{c < Y \leq d\}) = \int_c^d \int_a^b f_{X,Y}(x, y) \, dx \, dy.$
- For all B Borel subset of $\mathbb{R}^2$ (meaning that $\{X \in B\}$ is an event)

$$P((X, Y) \in B) = \iint_B f_{X,Y}(x, y) \, dx \, dy.$$

Joint distribution – visualization

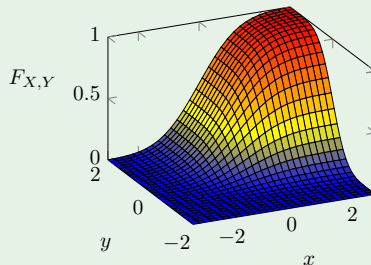
Example

Joint density



$$f_{X,Y}(x,y) = \frac{1}{\pi} e^{-\frac{x^2}{2} - 2y^2}$$

Joint distribution function



$$F_{X,Y}(x,y) = \int_{-\infty}^y \int_{-\infty}^x \frac{1}{\pi} e^{-\frac{u^2}{2} - 2v^2} du dv$$

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*Let X and Y be two random variables having a joint continuous distribution with joint density $f_{X,Y}$. Then X and Y are both continuous too, and the **marginal densities** f_X , f_Y are given by*

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \, dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \, dx.$$

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Proof

We know that:

$$F_X(x) = P(X \leq x) = P(X \leq x \cap Y \in \mathbb{R}) = \int_{-\infty}^x \left(\int_{-\infty}^{+\infty} f_{X,Y}(u, v) \, dv \right) du.$$

The statement of the theorem is obtained by differentiating with respect to x , or by comparing this formula to the definition of the distribution function of a continuous random variable. The second part is analogous. □

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Proof

Two random variables X and Y are independent if

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Taking the derivatives of both sides with respect to both x and y yields one implication. Integrating both sides of the equality for densities yields the other direction. □

Independence of continuous random variables

Remark

While verifying the independence of X and Y we can use the following.

Consequence: If it is possible to **decompose** $f_{X,Y}$ to

$$f_{X,Y}(x,y) = g(x) \cdot h(y), \quad \forall x, y \in \mathbb{R},$$

where $g(x)$ and $h(y)$ are non-negative functions, then the variables X and Y are **independent**.

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The statement of the consequence can be formulated for independence of a general random vector $X_1, \dots, X_n$ too.

Example – marginal distribution and independence

Example

Let X and Y random variables having the joint probability density

$$f_{X,Y}(x,y) = ye^{-2x} \quad \text{for } x \in [0, +\infty) \text{ and } y \in [0, 2].$$

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Independence:

$$ye^{-2x} = f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = 2e^{-2x} \cdot \frac{y}{2} = ye^{-2x}.$$

Yes, they are independent!

Discrete conditional distribution

Now we will study the **distribution** of a random variable X under the assumption that we **know** the value of the variable $Y = y$.

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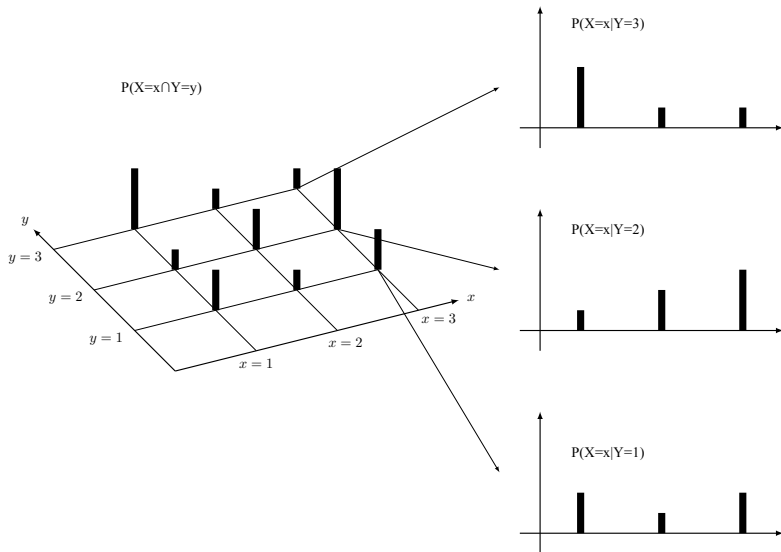
Let $P(Y = y) > 0$. Then, the **conditional distribution function** $F_{X|Y}(\cdot|y)$ of the variable X given $Y = y$ is defined as

$$F_{X|Y}(x|y) = P(X \leq x|Y = y).$$

The **conditional probabilities of values** of X given (under the condition of) $Y = y$ are given, analogously, by

$$P(X = x|Y = y).$$

Illustration of conditional probabilities $P(X = x|Y = y)$



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From the definition it follows that:

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Thus it holds that:

$$E(X|Y = y) = \sum_i x_i P(X = x_i|Y = y) = \sum_i x_i \frac{P(X = x_i \cap Y = y)}{P(Y = y)}.$$

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When observing two **continuous** random variables X and Y , **it is not possible to use an event $\{Y = y\}$ as a condition**, because $P(Y = y) = 0$.

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After taking a limit $\Delta y \rightarrow 0$ we intuitively obtain the result as

$$P(X \leq x \mid Y = y) = \int_{-\infty}^x \frac{f_{X,Y}(u, y)}{f_Y(y)} \, du.$$

Continuous conditional distribution

the previous inference lead us to the following formal definition:

Definition

The **conditional distribution function** of a variable X given (under the condition of) $Y = y$ is defined as

$$F_{X|Y}(x|y) = \int_{-\infty}^x \frac{f_{X,Y}(u, y)}{f_Y(y)} du,$$

for all y such that $f_Y(y) > 0$. We use the notation $P(X \leq x|Y = y) = F_{X|Y}(x|y)$, too.

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Analogously as in the discrete case we define the **conditional expectation** for continuous random variables:

Definition

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We compute the conditional expectation for a given value y as follows:

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x, y)}{f_Y(y)} dx = g(y),$$

where g is a function which arises from the integration.

Bayes' Theorem

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Let Y be continuous random variable. If X is continuous random variable too then it holds:

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$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t) dt}.$$

If X is discrete random variable than we have:

$$P(X = n|Y = y) = \frac{P(X = x) f_{Y|X}(y|x)}{\sum_k P(X = k) f_{Y|X}(y, k)}.$$

Recap

Joint distribution function of a random vector (X, Y) :

$$F_{X,Y}(x, y) = P(X \leq x \cap Y \leq y).$$

Discrete random variables X and Y

Joint probabilities of values:

$$P(X = x \cap Y = y)$$

Continuous random variables X and Y

Joint density:

$$f_{X,Y}(x, y)$$

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$$P(X = x) = \sum_{\text{all } y} P(X = x \cap Y = y)$$

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Independence of X and Y :

$$P(X = x \cap Y = y) = P(X = x) P(Y = y) \quad | \quad f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Conditional probabilities / density of X given $Y = y$:

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$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Conditional expectation of X given $Y = y$:

$$E(X|Y = y) = \sum_x x P(X = x|Y = y)$$

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