

Random variables III.

(Important discrete and continuous distributions)

Lecturer:
Francesco Dolce

Department of Applied Mathematics
Faculty of Information Technology
Czech Technical University in Prague

© 2011–2025 - Rudolf B. Blažek, Francesco Dolce, Roman Kotecký, Jitka Hrabáková, Petr Novák, Daniel Vařata

Probability and Statistics

BIE-PST, WS 2025/26, Lecture 5



Content

- **Probability theory:**

- ▶ Events, probability, conditional probability, Bayes' Theorem, independence of events.
- ▶ Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, **important discrete and continuous distributions**.
- ▶ Random vectors, joint and marginal distributions, independence of random variables, conditional distribution, functions of random vectors, covariance and correlation.
- ▶ Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.

- **Mathematical statistics:**

- ▶ Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
- ▶ Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

- A **random variable** X is a measurable function which assigns real values to the outcomes of a random experiment.
- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

$$F_X(x) = P(X \leq x).$$

- There are two major types of random variables:
 - ▶ **Discrete**, taking only countably many possible values.
 - ▶ **Continuous**, taking values from an interval.
- The distribution can be given by:
 - ▶ for discrete distributions by the **probabilities** of possible values $P(X = x_k)$.
 - ▶ for continuous distributions by the **density** f_X for which

$$F_X(x) = \int_{-\infty}^x f(t)dt.$$

Constant random variable

A constant random variable describes a non-random situation when we have only one possible result occurring with probability of 1.

Definition

A random variable X is called **constant**, if for some $c \in \mathbb{R}$ it holds that:

$$X(\omega) = c \text{ for all } \omega \in \Omega.$$

In other words it holds that:

$$P(X = c) = 1, \quad P(X = x) = 0 \quad \forall x \neq c.$$

We say that a constant random variable has a **deterministic** or **degenerate** distribution.

The **distribution function** of a constant random variable is

$$F_X(x) = \begin{cases} 0 & \text{for } x < c \\ 1 & \text{for } x \geq c. \end{cases}$$

Constant random variable – expectation, variance

$$P(X = c) = 1, \quad P(X = x) = 0 \quad \forall x \neq c$$

Expectation and variance:

$$\mathbf{E}(X) = \sum_{x_k} x_k \cdot P(X = x_k) = c \cdot P(x = c) = \mathbf{c}$$

$$\mathbf{var}(X) = E(X - E(X))^2 = E(X^2) - (E(X))^2 = c^2 - (c)^2 = \mathbf{0}.$$

In calculations we use:

$$E(c) = c \quad \text{– the center of mass of a constant } c \text{ is } c \text{ itself;}$$

$$\mathbf{var}(c) = 0 \quad \text{– the width of the graph with only one number } c \text{ is } 0.$$

Bernoulli (Alternative) distribution

Suppose we perform a random experiment with **two** possible **outcomes** (alternatives). We assign values 0 (failure) and 1 (success) to these outcomes. We can use for example one toss with an unbalanced coin.

Suppose that a success occurs with the probability p .

Definition

A random variable X has the **Bernoulli** (alternative) **distribution** with parameter $p \in [0, 1]$, if it holds that:

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

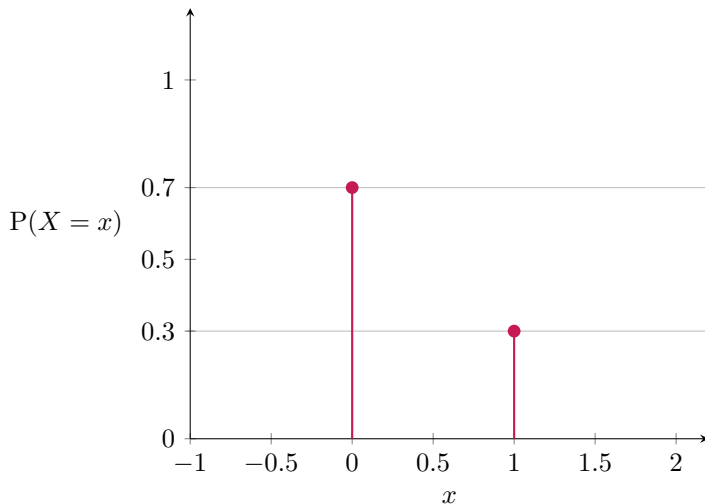
Notation: $X \sim \text{Be}(p)$ or $X \sim \text{Bernoulli}(p)$ or $X \sim \text{Alt}(p)$.

Example – toss with a coin

- Let us choose $X(\text{Heads}) = 1$ and $X(\text{Tails}) = 0$.
- We denote the occurrence of Heads as a success: $p = P(\text{Heads})$.

Bernoulli distribution – graph of probabilities

Probabilities of values of the Bernoulli distribution with $p = 0.3$:



Bernoulli distribution – expectation, variance

Bernoulli random variable:

$$P(X = 1) = p \in [0, 1] \quad (\text{Heads, success})$$

$$P(X = 0) = 1 - p \quad (\text{Tails, failure}).$$

Expectation and variance:

$$E(X) = \sum_{x_k} x_k P(X = x_k) = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$E(X^2) = \sum_{x_k} x_k^2 P(X = x_k) = 1^2 \cdot p + 0^2 \cdot (1 - p) = p$$

$$\text{var}(X) = E(X^2) - E(X)^2 = p - p^2 = p(1 - p).$$

Binomial distribution

If we repeat the coin tossing we can be interested in how many times from n tosses we have obtained Heads:

- Consider n independent experiments with two possible outcomes.
- Again suppose that we succeed in each experiment with probability p .
- The probability that exactly k out of n attempts ended with a success is

$$\binom{n}{k} p^k (1 - p)^{n-k}.$$

Definition

A random variable X has the **binomial distribution** with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$, if

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n.$$

Notation: $X \sim \text{Bin}(n, p)$, $X \sim \text{Binom}(n, p)$.

Binomial distribution – normalization

To prove that the binomial distribution is correctly defined, we verify the **normalization condition**, i.e., that the sum of all probabilities is equal to 1:

$$\sum_{k=0}^n P(X = k) = 1.$$

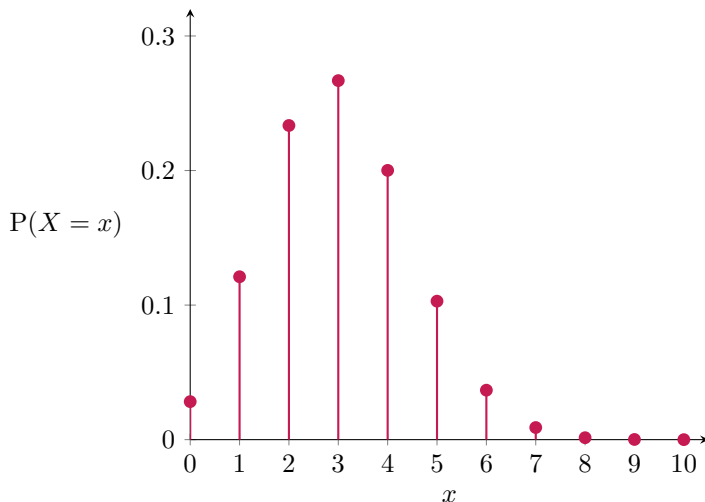
According to the **binomial theorem** it holds that

$$\sum_{k=0}^n P(X = k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1.$$



Binomial distribution – graph of probabilities

Binomial distribution with parameters $n = 10$ and $p = 0.3$:



Binomial distribution – expectation

Binomial random variable $X \sim \text{Binom}(n, p)$:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n.$$

$$E(X) = \sum_{k=0}^n k P(X = k) = \sum_{k=0}^n \binom{n}{k} k p^k (1 - p)^{n-k}.$$

The sum on the right hand side looks, except for a term $k p^k$, like

$$\sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} = (p + (1 - p))^n = 1^n = 1.$$

Notice that $(p^k)' = k p^{k-1}$ and thus $p(p^k)' = k p^k$.

After differentiating both sides with respect to p and multiplying by p we obtain the needed expression.

Binomial distribution – expectation

or

$$\begin{aligned}
 \mathbb{E}(X) &= \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} & / \quad k \binom{n}{k} = n \binom{n-1}{k-1} \\
 &= \sum_{k=1}^n n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)} \\
 &= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} & / \quad n-1 = m, \quad k-1 = h \\
 &= np \sum_{h=0}^m \binom{m}{h} p^h (1-p)^{m-h} \\
 &= np \cdot (p + (1-p))^m = np
 \end{aligned}$$

Binomial distribution – variance

Similarly we have:

$$\begin{aligned} E(X^2) &= \sum_{k=0}^n k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^n k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot n \binom{n-1}{k-1} p^k (1-p)^{n-k} = np \sum_{k=1}^n k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\ &= np \left(\sum_{k=1}^n (k-1) \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} + \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \right) \\ &= np((n-1)p + 1) \end{aligned}$$

Therefore

$$\text{var}(X) = E(X^2) - (E(X))^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p)$$

Indicator of an event

A special and important example of a Bernoulli random variable is the **indicator of an event**.

Definition

Let $A \in \mathcal{F}$ be an event. The random variable $\mathbb{1}_A : \Omega \rightarrow \{0, 1\}$ defined as

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

is called the **indicator** (or **characteristic function**) of the event A .

For the indicator of an event A it holds that:

$$\begin{aligned} p &= P(\mathbb{1}_A = 1) = P(A), \\ 1 - p &= P(\mathbb{1}_A = 0) = P(A^c) = 1 - P(A). \end{aligned}$$

Indicator of event – examples

Examples – tossing a coin

- The Bernoulli random variable X from the previous example (tossing a coin) is nothing but an indicator of the event $\{H\}$. Thus $X = \mathbb{1}_{\{H\}} = \mathbb{1}_H$.
- The Binomial random variable X corresponding to number of Heads in n tosses can be expressed as the sum

$$X = \sum_{i=1}^n \mathbb{1}_{H_i},$$

where $\mathbb{1}_{H_i}$ is the indicator of the event $H_i = \text{“Heads appears in the } i^{\text{th}} \text{ toss”}$.

Remark:

Expressing a binomial variable as a sum of (Bernoulli) indicators often leads to a significant simplification of calculations.

Geometric distribution

Another important event is the first occurrence of Heads in a sequence of coin tosses:

- Consider a **sequence of independent** experiments with **two** possible **outcomes**.
- Suppose that each experiment ends with a success with probability p .
- Probability that the **first successful** attempt the is k^{th} in the sequence is

$$(1 - p)^{k-1}p.$$

Definition

A random variable X has the **geometric distribution** with parameter $p \in (0, 1)$, if

$$P(X = k) = (1 - p)^{k-1}p, \quad k = 1, 2, \dots$$

Notation: $X \sim \text{Geom}(p)$.

Again we verify the **normalization condition**:

$$\sum_{k=1}^{\infty} P(X = k) = \sum_{k=1}^{\infty} (1 - p)^{k-1}p = p \sum_{k=0}^{\infty} (1 - p)^k = \frac{p}{1 - (1 - p)} = 1.$$

Geometric distribution – distribution function

The **distribution function** of the geometric distribution can be expressed as

$$\begin{aligned} F_X(k) = P(X \leq k) &= \sum_{i=1}^k p(1-p)^{i-1} = p \sum_{j=0}^{k-1} (1-p)^j \\ &= p \frac{1 - (1-p)^k}{1 - (1-p)} = 1 - (1-p)^k. \end{aligned}$$

For non-integer points $x > 0$ the value of distribution function is equal to value at point $\lfloor x \rfloor$ (the lower integer part of x):

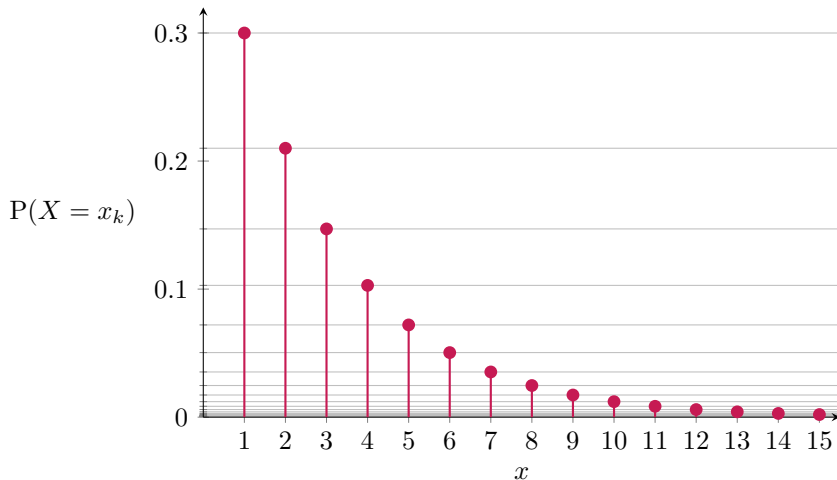
$$F_X(x) = F_X(\lfloor x \rfloor) = 1 - (1-p)^{\lfloor x \rfloor}.$$

The probability that the success does not occur after k attempts can be computed as

$$P(X > k) = (1-p)^k \quad \text{and thus} \quad F_X(k) = 1 - P(X > k) = 1 - (1-p)^k.$$

Geometric distribution – graph of probabilities

Geometric distribution with parameter $p = 0.3$:



Geometric distribution – expectation

$$P(X = k) = (1 - p)^{k-1}p \quad k = 1, 2, \dots$$

$$E(X) = \sum_{\text{all } x_k} x_k P(X = x_k) = \sum_{k=1}^{\infty} k (1 - p)^{k-1}p = p \sum_{k=1}^{\infty} k (1 - p)^{k-1}.$$

The sum on the right-hand side looks as the derivative of $-\sum_{k=0}^{\infty} (1 - p)^k$:

$$\begin{aligned} \textcolor{red}{E} X &= \sum_{k=1}^{\infty} k(1 - p)^{k-1}p = -p \left(\sum_{k=1}^{\infty} (1 - p)^k \right)' \\ &= -p \left(\frac{1}{1 - (1 - p)} \right)' = -p \left(\frac{-1}{p^2} \right) \\ &= \textcolor{red}{\frac{1}{p}}. \end{aligned}$$

Geometric distribution – variance

We can compute $E(X^2)$ using the same procedure. From the above we know that

$$\begin{aligned} E(X^2) &= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p = p \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} \\ &= p \left(\sum_{k=1}^{\infty} -k(1-p)^k \right)' = p \left((1-p) \sum_{k=1}^{\infty} -k(1-p)^{k-1} \right)' \\ &= p \left((1-p) \left(\sum_{k=1}^{\infty} (1-p)^k \right)' \right)' = p \left((1-p) \left(\frac{1}{p} \right)' \right)' \\ &= p \left(\frac{p-1}{p^2} \right)' = p \frac{p^2 - (p-1)2p}{p^4} = \frac{2-p}{p^2}. \end{aligned}$$

Thus

$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{2-p}{p^2} - \left(\frac{1}{p} \right)^2 = \frac{1-p}{p^2}.$$

Poisson distribution – motivation

The **number** of random occurrences during a **given time** is often modeled by the Poisson distribution:

- For example X = “number of server requests in 15 seconds”.
- Or X = “number of customers in a shop during lunch time”.
- Finite population: **n individuals independently** decide whether to go to a shop or not.
 - ▶ Then X is a **binomial** random variable: $X \sim \text{Binom}(n, p)$.
- Infinite population: we are interested in $X \sim \text{Binom}(n, p)$ **for $n \rightarrow \infty$** .
 - ▶ Useful approximation for great populations (molecules of gas, internet users, etc.).

Example – number of customers in a shop during lunch time

- number of inhabitants in a city: n ;
- number of shops proportional to the number of inhabitants: $n_{\text{shops}} = \rho n$, where ρ is the density of shops (number of shops per one inhabitant);
- probability that an inhabitant decides to go shopping: z ;
- probability that an inhabitant goes to a **particular** shop: $p = z/n_{\text{shops}} = z/(\rho n)$;
- number of inhabitants going to the particular shop: $X \sim \text{Binom}(n, p)$;
- expected value: $E X = np = nz/(\rho n) = z/\rho \quad \dots \text{constant.}$

Poisson distribution – motivation

Binomial distribution with $n \rightarrow \infty$, $p \rightarrow 0$ and $np = \lambda$ is

$$P(X = k) = \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

We **rearrange** the product and take a limit $n \rightarrow \infty$

$$P(X = k) = \frac{n}{n} \frac{(n-1)}{n} \dots \frac{(n-k+1)}{n} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$\begin{array}{ccccccc} \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & \dots & 1 & \frac{\lambda^k}{k!} & e^{-\lambda} & 1 \end{array}$$

Finally we have

$$\lim_{n \rightarrow \infty} P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Poisson distribution

Definition

A random variable X has the **Poisson distribution** with parameter $\lambda > 0$ if

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \dots$$

Notation: $X \sim \text{Poisson}(\lambda)$

Recalling the important formula:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

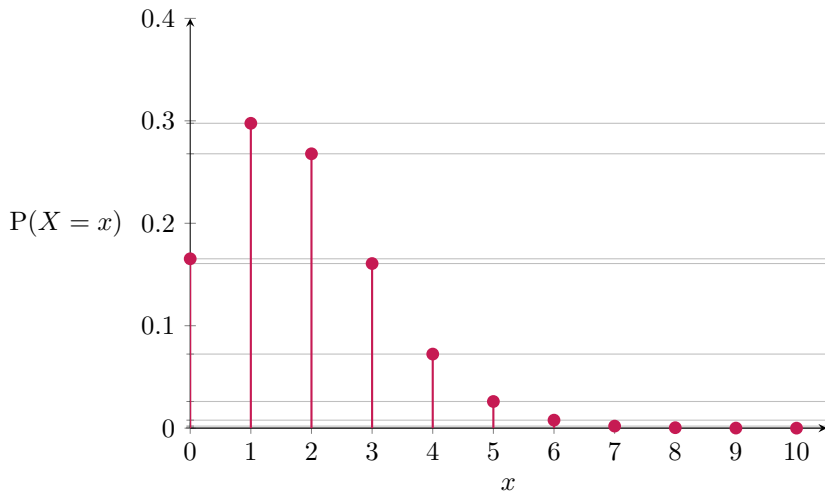
we can check that the normalization condition holds:

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1.$$



Poisson distribution – graph of probabilities

Poisson distribution with parameter $\lambda = 1.8$:



Poisson distribution – expectation

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \dots$$

The expectation is

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k P(X = k) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= \lambda e^{-\lambda} e^{\lambda} = \lambda. \end{aligned}$$

Poisson distribution – variance

$E(X^2)$ is computed similarly:

$$\begin{aligned} E(X^2) &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} k^2 \frac{\lambda^{k-1}}{k(k-1)!} \\ &= \lambda e^{-\lambda} \left(\sum_{k=1}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\ &= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \\ &= \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda}) = \lambda^2 + \lambda. \end{aligned}$$

Thus

$$\text{var}(X) = E(X^2) - (E X)^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda.$$

Recapitulation

- **Bernoulli** (Alternative) distribution with parameter p , $0 \leq p \leq 1$, $X \sim \text{Be}(p)$:
(other notations: $X \sim \text{Bernoulli}(p)$, $X \sim \text{Alt}(p)$)
(One toss with an unbalanced coin.)

$$P(1) = p, \quad P(0) = 1 - p \qquad E X = p, \quad \text{var } X = p(1 - p).$$

- **Binomial** distribution with parameters n and p , $0 \leq p \leq 1$, $X \sim \text{Binom}(n, p)$:
(Number of Heads in n tosses with an unbalanced coin.)

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \qquad E X = np, \quad \text{var } X = np(1 - p).$$

- **Geometric** distribution with parameter p , $0 < p < 1$, $X \sim \text{Geom}(p)$:
(Number of tosses with an unbalanced coin until first Heads appears.)

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, \dots \qquad E X = \frac{1}{p}, \quad \text{var } X = \frac{1 - p}{p^2}.$$

- **Poisson** distribution with parameter $\lambda > 0$, $X \sim \text{Poisson}(\lambda)$:
(Limit of the binomial distribution for $n \rightarrow \infty$.)

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \dots \qquad E X = \text{var } X = \lambda.$$

Uniform distribution

All values in some interval (a, b) can occur with “equal” probability.

Definition

A continuous random variable X has the **uniform** distribution with parameters $a < b$, $a, b \in \mathbb{R}$, if its density has the form:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in (a, b), \\ 0 & \text{elsewhere.} \end{cases}$$

Notation: $X \sim \text{Unif}(a, b)$, $X \sim \text{U}(a, b)$.

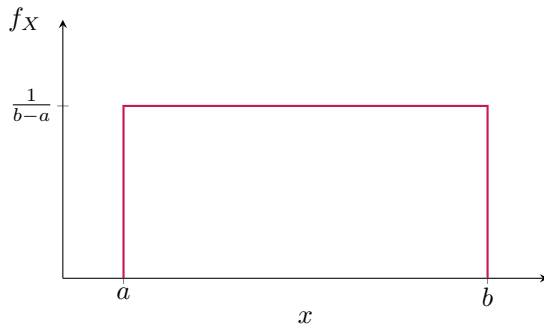
Normalization condition:

$$\int_{-\infty}^{+\infty} f_X(x) dx = \int_a^b \frac{1}{b-a} dx = \frac{b-a}{b-a} = 1.$$

Distribution function:

$$F_X(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{t}{b-a} \right]_a^x = \frac{x-a}{b-a} \quad \text{for } x \in [a, b].$$

Uniform distribution – graph of density



Uniform distribution – expectation, variance

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in (a, b), \\ 0 & \text{elsewhere.} \end{cases}$$

$$\mathbf{E}(X) = \int_a^b x f_X(x) \, dx = \int_a^b \frac{x}{b-a} \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{a+b}{2},$$

$$\mathbf{E}(X^2) = \int_a^b x^2 f_X(x) \, dx = \int_a^b \frac{x^2}{b-a} \, dx = \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b = \frac{a^2 + ab + b^2}{3},$$

$$\mathbf{var}(X) = \mathbf{E}(X^2) - (\mathbf{E} X)^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

Exponential distribution

Very often used in queuing theory and theory of random processes.

Definition

A random variable X has the **exponential** distribution with parameter $\lambda > 0$, if its density has the form:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \in [0, +\infty), \\ 0 & \text{elsewhere.} \end{cases}$$

Notation: $X \sim \text{Exp}(\lambda)$.

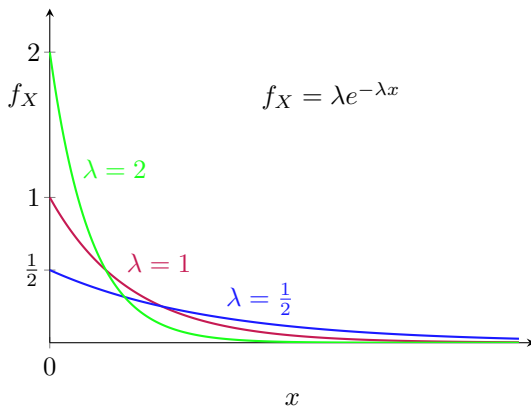
Normalization:

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = [-e^{-\lambda x}]_0^{+\infty} = 0 - (-1) = 1.$$

Distribution function:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = [-e^{-\lambda t}]_0^x = 1 - e^{-\lambda x}.$$

Exponential distribution – graph of density



Exponential distribution – expectation, variance

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

$$\mathbb{E}(X) = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx \stackrel{\text{by parts}}{=} \frac{1}{\lambda}$$

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2 f_X(x) dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx \stackrel{2 \times \text{by parts}}{=} \frac{2}{\lambda^2}$$

$$\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E} X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

✓ Details during tutorials.

Normal distribution

The normal distribution occurs in nature (population lengths, weights, etc.) and is used as an approximation for sums and means of random variables.

Definition

A random variable X has the **normal** (Gaussian) distribution with parameters μ and $\sigma^2 > 0$, if the density has the form:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in (-\infty, +\infty).$$

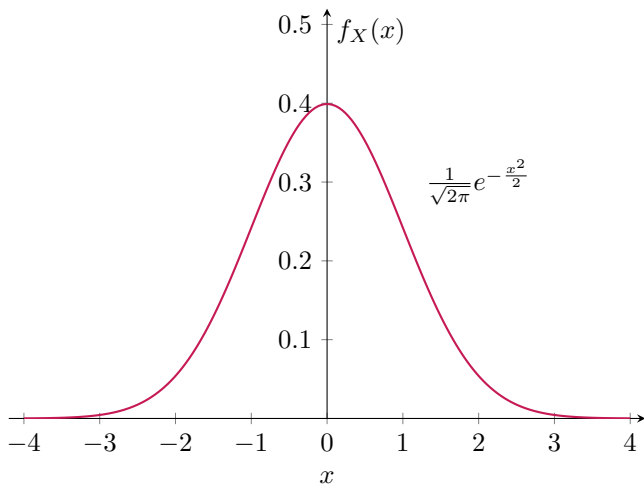
Notation: $X \sim N(\mu, \sigma^2)$.

- Attention: Some literature and software uses $X \sim N(\mu, \sigma)$.
- We will further use the symbol σ for $\sqrt{\sigma^2}$.
- $N(0, 1)$ is called the **standard normal** distribution.

Distribution function: cannot be given explicitly, only numerically. The standard normal distribution function is tabulated and denoted as Φ .

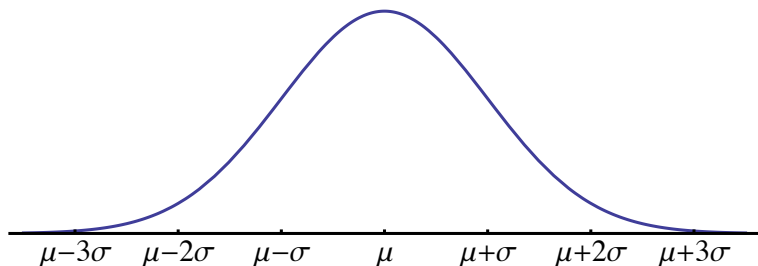
$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Standard normal distribution $N(0, 1)$



$$\Phi(-x) = 1 - \Phi(x)$$

Density of the normal distribution: $X \sim N(\mu, \sigma^2)$

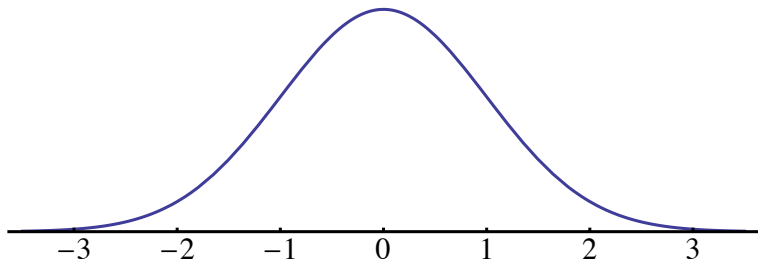


$$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$$

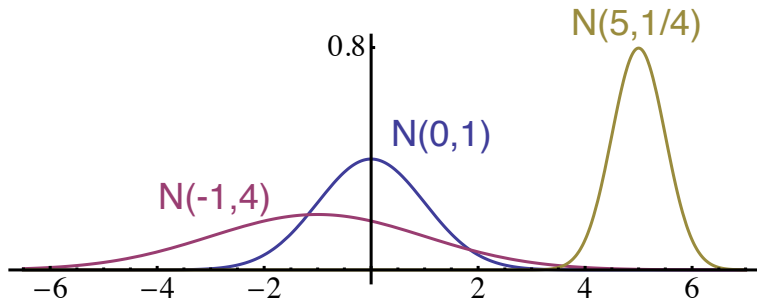
$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$$

$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.997$$

Density of the normal distribution: $Z \sim N(0, 1)$



Density of the normal distribution



Normal distribution – expectation, variance

Normal random variable $X \sim N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in (-\infty, +\infty).$$

$$E(X) = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \stackrel{\text{substitution}}{=} \mu.$$

$$\text{var}(X) = \sigma^2.$$

Standardization of random variable

Consider a random variable X with expected value $E X = \mu$ and variance $\text{var } X = \sigma^2$.

In the easiest possible way, try to **convert** the variable X to the variable Z with parameters $E Z = 0$ and $\text{var } Z = 1$ (**standardization**):

- We subtract the expectation μ :

$$E(X - \mu) = E X - \mu = 0 \quad \text{and} \quad \text{var}(X - \mu) = \text{var } X = \sigma^2.$$

- We rescale with the value $\sigma = \sqrt{\text{var } X}$:

$$E\left(\frac{X - \mu}{\sigma}\right) = \frac{E(X - \mu)}{\sigma} = 0 \quad \text{and} \quad \text{var}\left(\frac{X - \mu}{\sigma}\right) = \frac{\text{var}(X - \mu)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$$

The required transformation is thus **linear** and the random variable

$$Z = \frac{X - \mu}{\sigma}$$

indeed has a zero mean and a variance of 1.

Standardization of a normal random variable

For practical uses we are interested in the standardization of the normal random variable.

Theorem

Let a random variable X have the normal distribution $X \sim N(\mu, \sigma^2)$. Then the random variable

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution, $Z \sim N(0, 1)$.

Proof

$$F_Z(z) = P(Z \leq z) = P\left(\frac{X - \mu}{\sigma} \leq z\right) = P(X \leq \sigma z + \mu) = F_X(\sigma z + \mu)$$

$$f_Z(z) = \frac{\partial F_Z}{\partial z}(z) = \frac{\partial F_X}{\partial z}(\sigma z + \mu) = \sigma f_X(\sigma z + \mu)$$

$$= \sigma \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\sigma z + \mu - \mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

□

Standardization of a normal random variable

Remark

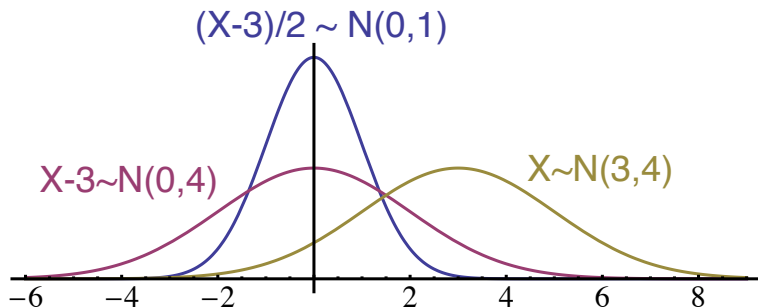
From the previous theorem it follows that:

If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$.

This is used for obtaining the values of the distribution function of the variable X from the tables of the standard normal distribution Z :

$$\begin{aligned} F_X(x) &= P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\ &= P\left(Z \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right). \end{aligned}$$

Standardization of a normal random variable



Recapitulation

- **Uniform** distribution on the interval $[a, b]$, $X \sim \text{Unif}(a, b)$ or $X \sim \text{U}(a, b)$:

$$f_X(x) = \frac{1}{b-a}, \quad x \in [a, b] \qquad \mathbb{E} X = \frac{a+b}{2}, \quad \text{var } X = \frac{(b-a)^2}{12}.$$

- **Exponential** distribution with parameter $\lambda > 0$, $X \sim \text{Exp}(\lambda)$:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \in [0, +\infty) \qquad \mathbb{E} X = \frac{1}{\lambda}, \quad \text{var } X = \frac{1}{\lambda^2}.$$

- **Normal (Gaussian)** distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, $X \sim \text{N}(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, +\infty) \qquad \mathbb{E} X = \mu, \quad \text{var } X = \sigma^2.$$