
REVIEW - RANDOM VARIABLES II: MOMENTS

Basic characteristics of random variables

- The **expectation** (mean value) $E X$, giving the mean (expected) value of X .
- The **variance** $\text{var } X$, giving the dispersion (variability) of X around its mean $E X$. The variance is defined as

$$\text{var } X = E(X - E X)^2 = E X^2 - (E X)^2$$

and is always **non-negative**.

Discrete random variables:

For a discrete random variable with values $x_1, x_2, \dots$:

- The expectation of X is computed as

$$E X = \sum_k x_k P(X = x_k) \quad (\text{if the sum exists});$$

- The variance of X is computed as

$$\text{var } X = E X^2 - (E X)^2 = \sum_k x_k^2 P(X = x_k) - \left(\sum_k x_k P(X = x_k) \right)^2 \quad (\text{if both sums exist});$$

- The expectation of the random variable $Y = h(X)$ is computed as

$$E Y = E h(X) = \sum_k h(x_k) P(X = x_k) \quad (\text{if the sum exists}),$$

or directly out of its distribution $E Y = \sum_y y P(Y = y)$.

Continuous random variables:

- The expectation of a continuous random variable X is computed as

$$E X = \int_{-\infty}^{\infty} x f_X(x) dx \quad (\text{if the integral exists});$$

- the expectation of $Y = h(X)$ is computed as

$$E h(X) = \int_{-\infty}^{\infty} h(x) f(x) dx;$$

- in particular

$$E X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx.$$

Useful properties

- The moment generating function of a random variable X is a function of $t \in \mathbb{R}$ and is defined as $M(t) = E e^{tX}$. It holds that:

$$E X = M'(0), \quad \text{and} \quad \text{var } X = M''(0) - (M'(0))^2;$$

- when $a, b \in \mathbb{R}$ and X is a random variable, then

$$E(a + bX) = a + b E X, \quad \text{and} \quad \text{var}(a + bX) = b^2 \text{var } X;$$

- when $a, b \in \mathbb{R}$ and X and Y are random variables, then

$$E(aX + bY) = a E X + b E Y.$$

EXERCISES 4 - RANDOM VARIABLES II: MOMENTS

1. Find the expected value and the variance of the random variable X denoting the number of Heads appearing after tossing two coins.

Remark: This random variable can be represented as a sum of the results of the tosses, where Heads is represented as 1, Tails as 0 and $P(\text{Heads}) = p$.

2. We are rolling a six-sided die until a 6 occurs. Denote the moment when a 6 occurs for the first time as the X^{th} attempt. Find the expected number of rolls needed to get a six.

3. Suppose we are rolling a balanced n -sided die with values 1 to n . Let X be the number of points landed (*discrete uniform distribution*).

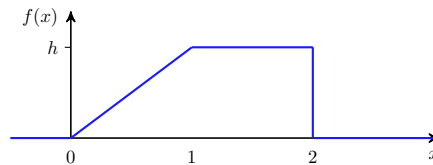
a) What is the expectation of X ?

b) Is it possible to establish such a uniform distribution on values 1 to infinity?

4. Find the expected value and the variance of the random variable with probability density given by the following formula:

$$f_X(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } x \in (0, 1] \\ 2 - x & \text{for } x \in (1, 2] \\ 0 & \text{for } x > 2. \end{cases}$$

5. Let X be a continuous random devariable defined in Exercise 4 of Tutorial 3. Its density is given by the graph:



Find the expected value and the variance of X .

6. Suppose we observe a random variable X with $E X = 1$ and $\text{var } X = 2$. Find $E(3X - 4)$ and $\text{var}(2X + 1)$.

7. Let X be a random variable taking values 0, 1, 2, 3, 4, 5, 6, 7 with the same probability. We define the random variable Y as

$$Y = \begin{cases} 1 & \text{for } X < \frac{1}{2} \\ 2 & \text{for } X \geq \frac{1}{2}. \end{cases}$$

Find the expected value and the variance of Y .

8. Suppose that a random variable X has the following properties:

$$E X = 0, \quad E(X^2) = 1, \quad E(X^3) = 0, \quad E(X^4) = 2.$$

Random variables Y and Z are defined as

$$Y = 1 + X^2, \quad \text{and} \quad Z = 1 - X.$$

Find the expected values and variances of Y and Z .

ADDITIONAL EXERCISES - RANDOM VARIABLES II: MOMENTS

Discrete random variables

9. We have two 500 CZK banknotes, one 1000 CZK and one 2000 CZK in our pocket. A pickpocket reaches into the pocket and steals two banknotes at random. Let X be the random variable denoting the total value of our lost money.

- a) Find the distribution of X .
- b) Calculate the expected (mean) loss.
- c) Calculate the variance of X .
- d) Plot the distribution function of X .

10. Consider a lottery where a lottery ticket is a winning one with a probability p and a losing one with $1 - p$. We devised a strategy of buying the tickets until we win.

- a) Determine the distribution and expected number of losing tickets bought until we win.
- b) Suppose that the winning award is 100,000 CZK and one ticket costs 100 CZK. What is the minimum needed value of p for our strategy to pay off?

11. There are n gentlemen who left their hats at the cloakroom in a theater. After the play ends, the absent-minded cloakroom keeper gives each gentleman a hat selected at random. What is the expected number of gentlemen who got their own hats?

Transformations of random variables

12. Let X follow the Cauchy distribution with density

$$f_X(x) = c \cdot \frac{1}{1 + x^2}, \quad x \in \mathbb{R}.$$

- a) Find c so that f_X truly is a density of a random variable.
- b) Find the expectation EX .
- c) Find the density of $Y = 1/X$.

13. Suppose that the radius of a soap bubble in centimeters is a uniformly distributed random variable on the interval $[0, 1]$. What is the distribution function, density and expectation of the volume of the bubble?

Quantile function

14. The time spent waiting for a tram forms a random variable X with the density

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- a) Find the quantile function F^{-1} of X and plot its graph.
- b) Find the median of X and compare it to the expectation EX .
- c) Suppose U is a random variable uniformly distributed on the interval $[0, 1]$. Find the distribution of $Y = F^{-1}(U)$.