# Review - Random vectors I: independence, conditionals

**Joint distribution:**
If $(X, Y)$ has a discrete distribution with values $(x_i, y_j)$, the joint distribution is given by the probabilities $\mathrm{P}(X = x_i \cap Y = y_j)$ for all possible $(x_i, y_j)$.

If $(X, Y)$ is continuous with a density $f(x, y)$, then the joint distribution is given by the joint density $f_{XY}(x, y)$.

**Marginal distribution:**
If $(X, Y)$ has a discrete distribution with values $(x_i, y_j)$, then the marginal distribution of $X$ is discrete and can be computed as

$$\mathrm{P}(X = x_i) = \sum_j \mathrm{P}(X = x_i \cap Y = y_j).$$

If $(X, Y)$ is continuous with a density $f(x, y)$, then the marginal density of $X$ is computed as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Similarly for the marginal distribution of $Y$.

**Independence:**
Two random variables $X$ and $Y$ are independent, if

$$\mathrm{P}(X \le x \cap Y \le y) = \mathrm{P}(X \le x) \cdot \mathrm{P}(Y \le y) \quad \forall\, (x, y) \in \mathbb{R}^2.$$

If $(X, Y)$ has a continuous distribution then $X$ and $Y$ are independent if and only if $f_{XY}(x, y) = f_X(x) f_Y(y)$ for almost all $(x, y) \in \mathbb{R}^2$.

If $(X, Y)$ has a discrete distribution then $X$ and $Y$ are independent if and only if $\mathrm{P}(X = x_i \cap Y = y_j) = \mathrm{P}(X = x_i)\, \mathrm{P}(Y = y_j)$ for all possible $(x_i, y_j)$.

**Conditional distribution:**

a) If $(X, Y)$ has a discrete distribution with values $(x_i, y_j)$, then the conditional distribution of $X$ given $Y = y$ is discrete and can be computed as

$$\mathrm{P}(X = x_i | Y = y_j) = \frac{\mathrm{P}(X = x_i \cap Y = y_j)}{\mathrm{P}(Y = y_j)}.$$

b) If $(X, Y)$ is continuous with a density $f(x, y)$, then the conditional density of $X$ given $Y = y$ is computed as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

**Conditional expectation:**
If $(X, Y)$ has a discrete distribution with values $(x_i, y_j)$, then the conditional expectation of $X$ given $Y = y$ is defined as

$$\mathrm{E}(X | Y = y_j) = \sum_{x_i} x_i\, \mathrm{P}(X = x_i | Y = y_j).$$

If $(X, Y)$ has a continuous distribution, then the conditional expectation of $X$ given $Y = y$ is defined as

$$\mathrm{E}(X | Y = y) = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\, \mathrm{d}x.$$

# Exercises 6 - Random vectors I: independence, conditionals

**1.** Suppose that a random vector $(X, Y)$ has the following joint probability distribution:

| $\mathrm{P}(X = x \cap Y = y)$ | | Y: 0 | 1 | 2 |
|---|---|---|---|---|
| X | 1 | 0.4 | 0.15 | 0.05 |
| | 2 | 0.3 | 0.06 | 0.04 |

a) Find the marginal distributions of the random variables $X$ and $Y$.

b) Are $X$ and $Y$ independent?

c) Find the conditional distribution of $X|Y = y$.

d) Find $\mathrm{E}(X|Y = y)$.

e) Find the conditional distribution of $X + Y$ given $Y = y$.

**2.** Consider two continuous random variables $X$ and $Y$ with joint density

$$f(x, y) = \begin{cases} \frac{1}{5}(4xy + 4x - 8y) & \text{for } 1 \le x \le 2 \text{ and } 0 \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are $X$ and $Y$ independent?

b) Find the conditional density $f_{Y|X}(y|x)$.

c) Find $\mathrm{E}(Y|X = x)$.

d) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

**3.** Consider two continuous random variables $X$ and $Y$ with the joint density

$$f(x, y) = \begin{cases} \frac{1}{2} y e^{(-x-y^2)/2} & \text{for } x > 0 \text{ and } y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are $X$ and $Y$ independent?

b) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

c) Find $\mathrm{E}(XY)$. *(See next Tutorial)*

**4.** Let $X$ and $Y$ be two continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{1}{3}(-4xy - 4x + 8y + 8) & \text{for } 1 \le x \le 2 \text{ and } 0 \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are $X$ and $Y$ independent?

b) Find $\mathrm{E}(X + Y)$. *(See next Tutorial)*

c) Find $\mathrm{E}(XY)$. *(See next Tutorial)*

**5.** Let $X$ and $Y$ be continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{3}{8} x y^2 & \text{for } 0 \le x \le 1, \text{ and } -2 \le y \le 2\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the marginal density of the random variable $X$.

b) Find the conditional density $f_{Y|X}(y|x)$.

**6.** Suppose we roll two dice. Let $X$ be the random variable defined as the sum of both rolled numbers and $Y$ the random variable is defined as the difference between the number rolled on the first and the number rolled on the second dice. (i.e., $Y = 1^{st} - 2^{nd}$). Find $\mathrm{E}(X|Y = 2)$.

## ADDITIONAL EXERCISES - RANDOM VECTORS I: INDEPENDENCE, CONDITIONALS

**7.** Two people are betting on the result of one roll of a six sided die. If a person is correct in their bet, they win 1. If the person is not correct, they lose 1. Suppose that person X bets on even numbers and person Y bets on large numbers $\{4, 5, 6\}$. Let $X$ denote a random variable marking the winnings/losings of $X$ after one game, same for $Y$.

a) Find the joint distribution of $X$ and $Y$.

b) Find the marginal distributions for $X$ and $Y$.

c) Are $X$ and $Y$ independent?

d) Find the conditional distribution of $X$ given $Y = 1$ and given $Y = -1$.

e) Find the covariance and the coefficient of correlation between $X$ and $Y$. *(See next Tutorial)*