

BIE-DML - Discrete Mathematics and Logic

Tutorial 3

Mathematical Induction

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3.1 Proof Techniques

Exercise 3.1. Prove the statement below using direct proof:

"Let m, n, p be integers. If $m + n$ and $n + p$ are even then $m + p$ is even."

Remark 3.1. For two integers a and b , we say that a **divides** b (or that b is **divisible** by a) if there is an integer k such that $b = k \cdot a$. This fact is denoted by $a \mid b$.

Exercise 3.2. Prove this statement:

"Let a, b, c be positive integers. If a divides b and b divides c then a divides $b + c$."

Hint: use direct proof.

Exercise 3.3. Prove the statement below using indirect proof:

"Let m, n, p be positive integers. If $p = m \cdot n$ then $m \leq \sqrt{p}$ or $n \leq \sqrt{p}$."

Exercise 3.4. Prove the statement below using proof by contradiction:

"The sum of an irrational number and a rational number is irrational."

Exercise 3.5. Prove the statement below using proof by cases:

"For any real numbers a, b , $|a| \cdot |b| = |a \cdot b|$."

Exercise 3.6. Prove that for any propositional formulas E, F :

" $E \models F$ if and only if $E \wedge \neg F \models \perp$."

3.2 Weak induction

Exercise 3.7. Use mathematical induction to prove the following divisibility relations for every non-negative integer $n \in \mathbb{N}_0$:

- | | |
|--------------------------|--|
| a) $2 \mid (n^2 - n)$, | d) $2 \mid (n^2 + n)$, |
| b) $3 \mid (n^3 + 2n)$, | e) $5 \mid (n^5 - n)$, |
| c) $6 \mid (n^3 - n)$, | f) $21 \mid (4^{n+1} + 5^{2n-1})$ (for all $n \geq 1$). |

Exercise 3.8. Use mathematical induction to prove the following properties:

- a) $7^n - 1$ is divisible by 6 for every $n \in \mathbb{N}$.
- b) $11^n - 6$ is divisible by 5 for every $n \in \mathbb{N}$.
- c) $6 \cdot 7^n - 2 \cdot 3^n$ is divisible by 4 for every $n \in \mathbb{N}$.
- d) $3^n + 7^n - 2$ is divisible by 8 for every $n \in \mathbb{N}$.

Remark 3.2. In all the proofs below, if proving equality, we will mark the use of the inductive hypothesis with $\stackrel{IH}{=}$ (we will use $\stackrel{IH}{\leq}$, resp. $\stackrel{IH}{<}$, when proving (strict) inequality).

Exercise 3.9. Use mathematical induction to prove the following equalities:

- a) $\sum_{k=1}^n k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$
- b) $\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$
- c) $\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$
- d) $\sum_{k=1}^n (-1)^k k = -1 + 2 - 3 + \cdots + (-1)^n n = -\frac{1}{4} + (-1)^n \cdot \frac{(2n+1)}{4}.$

Exercise 3.10. Prove the following statements for every $n \in \mathbb{N}$ using the weak principle of mathematical induction.

- a) $\sum_{k=1}^n (2k-1) = 1 + 3 + 5 + \cdots + (2n-1) = n^2;$
- b) $n < 2^n;$
- c) $2^{n-1} \leq n!;$
- d) $\prod_{k=1}^n k = 1 \cdot 2 \cdot 3 \cdots n \leq n^n;$
- e) $\prod_{k=1}^n (2k)! = 2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n+1)!)^n;$
- f) $\sum_{k=1}^n \frac{1}{k!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 2 - \frac{1}{n!};$
- g) $\sum_{k=1}^n \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n};$
- h) $n^2 \leq 2^n$, for $n \geq 4;$
- i) $\sum_{k=1}^n \frac{1}{(2k-1)(2k+1)} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1};$
- j) $\sum_{k=1}^n k \cdot k! = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$

Exercise 3.11 (*). A harmonic sequence is a sequence of (rational) numbers

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Prove the following properties for every $n \in \mathbb{N}$:

- a) $H(1) + H(2) + \cdots + H(n) = (n+1) \cdot H(n) - n;$
- b) $H(2^n) \leq 1 + n;$

c) $H(2^n) \geq 1 + \frac{n}{2}$.

Hint : $\frac{1}{2^n + 1} \geq \frac{1}{2^{n+1}}, \quad \frac{1}{2^n + 2} \geq \frac{1}{2^{n+1}}, \quad \dots, \quad \frac{1}{2^n + 2^n} \geq \frac{1}{2^{n+1}}.$

Exercise 3.12. Consider the matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. Prove that $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for any integer $n \geq 1$.

Exercise 3.13. Prove that

a) $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for any $n \geq 1$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

b) $A^n = \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix}$ for any $n \geq 1$, where $A = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$, with $\varphi \in \mathbb{R}$.

Exercise 3.14. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that $A^n = \begin{pmatrix} f(n+1) & f(n) \\ f(n) & f(n-1) \end{pmatrix}$ for any integer $n \geq 1$, where $f(n)$ denotes the n^{th} Fibonacci number defined in Exercise 3.15.

3.3 Strong induction

Exercise 3.15. The Fibonacci sequence consists of numbers defined as follows:

- $f(0) = 0, \quad f(1) = 1,$
- $f(n) = f(n-1) + f(n-2) \quad \text{for } n \geq 2.$

Prove the following properties:

- a) $f(n) > \alpha^{n-2}$ for every $n \geq 3$, where $\alpha = \frac{1+\sqrt{5}}{2}$ (it is called the *golden ratio*);
- b) $f(n-1) \cdot f(n+1) = f^2(n) + (-1)^n$ for every $n \geq 1$;
- c) $\sum_{i=0}^n f^2(i) = f(n) \cdot f(n+1).$

Exercise 3.16. Consider an infinite sequence of integers $(A_0, A_1, A_2, A_3, \dots)$, whose members are defined recursively as follows:

$$A_{n+3} = A_{n+2} + 5 \cdot A_{n+1} + 3 \cdot A_n, \quad \forall n \in \mathbb{N}_0$$

and $A_0 = 1, A_1 = -2$, and $A_2 = 3$. Prove by mathematical induction that for all $n \in \mathbb{N}_0$,

$$A_n = (-1)^n(n+1).$$

3.4 Structural induction

Exercise 3.17. Consider a set S defined inductively using the rules below:

(R0) $1 \in S$,

(R1) If $n \in S$ then $3n + 2 \in S$ and $n^2 \in S$.

Use structural induction to prove that $\forall n \in S, 4|(n - 1)$.

Exercise 3.18. Consider the inductively defined set S of integers:

(R0) $6, 8 \in S$,

(R1) if $m, n \in S$, then $m + n \in S, m - n \in S$,

and all numbers in S are obtained by a finite application of these rules.

Find a property P which describes the elements of S and prove your guess by structural induction.

Exercise 3.19. Let $\mathbb{R}^2$ be a vector space and let M be a set of vectors defined inductively:

(R0) $(1, 0) \in M, (0, -1) \in M, (1, 1) \in M$;

(R1) $\forall \mathbf{a}, \mathbf{b} \in M : \mathbf{a} + \mathbf{b} \in M$ and $\forall \alpha \in \mathbb{R}, \forall \mathbf{a} \in M : \alpha \cdot \mathbf{a} \in M$.

Prove that M is a subset of the linear span of vectors $(1, 0), (0, -1)$, i.e., $M \subseteq \langle (1, 0), (0, -1) \rangle$.

Exercise 3.20. Consider all formulas of propositional logic:

(R0) elementary formulas are formulas of propositional logic;

(R1) if A, B are formulas then also $\neg(A), (A \wedge B), (A \vee B), (A \Rightarrow B), (A \Leftrightarrow B)$ are formulas.

Every formula can be obtained by a finite number of applications of rules (R0) and (R1).

Use **structural induction** to prove that every formula contains an equal number of left and right parentheses.

Exercise 3.21. Prove, using structural induction, that $\{\neg, \wedge\}$ is a complete system.

3.5 But be careful!

Exercise 3.22. Are the arguments presented below correct? If not, find the mistake.

a) For every nonzero integer a and for every $n \in \mathbb{N}_0$ we have $a^n = 1$ because

1. for $n = 0$: $a^0 = 1$;

2. assuming $a^n = 1$ for $n \geq 0$, then $a^{n+1} = a^n \cdot \frac{a^n}{a^{n-1}} = 1 \cdot \frac{1}{1} = 1$.

b) Assume $P(n)$ is the property stating that $1 + 2 + 3 + \dots + n = \frac{(n-1)(n+2)}{2}$. The implication $P(n) \Rightarrow P(n+1)$ is true (check!). Thus we have shown $P(n)$ is true for all $n \in \mathbb{N}$.

c) Assume $P(n)$ is the property stating that $2 + 4 + 6 + \dots + 2n = n^2 + n - 13$. The implication $P(n) \Rightarrow P(n+1)$ is true (check!). Thus we have shown $P(n)$ is true for all $n \in \mathbb{N}$.