BIE-DML - Discrete Mathematics and Logic

Tutorial 4

Sets and mappings

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4.1 Sets

Exercise 4.1. Consider the sets X, Y, Z, subsets of the universe $\mathcal{U}$, and their Cartesian products, subsets of $\mathcal{U} \times \mathcal{U}$. Decide whether the following statements are true or not. Prove it, or find a counterexample.

a) $X \cap (Y \setminus Z) = (X \cap Y) \setminus (X \cap Z)$,

b) $X \setminus (Y \cup Z) = (X \setminus Y) \cup Z$,

c) $\overline{X \setminus Y} = \overline{Y \setminus X}$,

d) $\overline{X \cap Y} \subseteq X$,

e) $(X \cap Y) \cup (Y \setminus X) = X$,

f) $(X \setminus Y) \cap (Y \setminus X) = \emptyset$,

g) $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$,

h) $X \times (Y \setminus Z) = (X \times Y) \setminus (X \times Z)$,

i) $\overline{(X \times Y)} = \overline{X} \times \overline{Y}$,

j) $X \setminus (Y \times Z) = (X \setminus Y) \times (X \setminus Z)$,

k) $X \cap (Y \times Z) = (X \cap Y) \times (X \cap Z)$,

l) $X \times \emptyset = \emptyset$.

Exercise 4.2. Let A, B, C be some sets in universe $\mathcal{U}$. Decide if the statements below are true or false. Prove or find a counterexample.

a) $A \setminus B \subseteq A$

b) $A \setminus B = A \cap \overline{B}$

c) $A \cap (B \setminus A) = \emptyset$

d) $(A \setminus B) \cap (B \setminus C) = \emptyset$

e) $(A \setminus B) \setminus C \subseteq A \setminus C$

f) $A \cup (B \setminus A) = A \cup B$

g) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

h) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

i) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

j) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

k) $A \subseteq B$ if and only if $B \subseteq A$

l) $A \subseteq B$ if and only if $A \cap B = A$

m) $A \subseteq B$ if and only if $A \cup B = B$

Exercise 4.3. Consider the following sets of positive integers defined for every index $i \in \mathbb{N}$ as follows:

$$M_i = \{1, 2, \ldots, i\}, \quad N_i = \{i, i+1, i+2, \ldots\}.$$

Determine what the following sets look like:

a) $\bigcup_{i=1}^{n} M_i$, $\bigcup_{i=1}^{\infty} M_i$,

b) $\bigcap_{i=1}^{n} M_i$, $\bigcap_{i=1}^{\infty} M_i$,

c) $\bigcup_{i=1}^{n} N_i$, $\bigcup_{i=1}^{\infty} N_i$,

d) $\bigcap_{i=1}^{n} N_i$, $\bigcap_{i=1}^{\infty} N_i$.

Compute the cardinality of all sets above.

4.2 Mappings

Exercise 4.4. Find the domain $\mathfrak{D}(f)$ and the image $\mathfrak{Im}(f)$ of the following mappings.

a) The mapping f which to every non-negative integer assigns the last digit of its decimal notation.

b) The mapping f which assigns the successor to every non-negative integer.

c) The mapping f which to every binary string assigns its length.

d) The mapping f which to every binary string assigns the number of factors '`01`' included in the string.

e) The mapping f which to every binary string assigns the number of occurrences of `1` minus the number of occurrences of `0` within the string.

f) The mapping f which to every integer k assigns the least square greater than or equal to k.

g) The mapping f which to every pair of real numbers assigns the bigger one of the two.

Exercise 4.5. Decide whether the following mappings are surjective and/or injective. Prove your answer.

a) $f(n) = n + 1, \quad f : \mathbb{Z} \to \mathbb{Z}$.

b) $f(n) = n + 1, \quad f : \mathbb{N}_0 \to \mathbb{N}_0$.

c) $f(n) = 13n, \quad f : \mathbb{Z} \to \mathbb{Z}$.

d) $f(n) = n^3, \quad f : \mathbb{Z} \to \mathbb{Z}$.

e) $f(x) = x^3, \quad f : \mathbb{R} \to \mathbb{R}$.

f) $f(n) = n^2 + 1, \quad f : \mathbb{Z} \to \mathbb{Z}$.

g) $f(n) = (-1)^n \cdot n, \quad f : \mathbb{N}_0 \to \mathbb{Z}$.

h) $f(n) = (n + 1, 2n), \quad f : \mathbb{N}_0 \to \mathbb{N}_0^2$.

i) $f(m, n) = m - n, \quad f : \mathbb{N}_0 \times \mathbb{N} \to \mathbb{Z}$.

Exercise 4.6. Prove or disprove the statements below.

a) If $f \circ g$ is surjective, then f must be surjective. Must g be surjective as well?

b) If $f \circ g$ is injective, then g must be injective. Must f be injective as well?

Exercise 4.7. Decide whether the following mappings are surjective and/or injective. Prove or provide a counterexample.

a) $f(m, n) = (m, m \cdot n), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^2$;

b) $f(x, y) = (x + y, x - y), \quad f : \mathbb{R}^2 \to \mathbb{R}^2$;

c) $f(x, y) = (x, x \cdot y), \quad f : \mathbb{R}^2 \to \mathbb{R}^2$;

d) $f(x, y) = (x^3, y^3), \quad f : \mathbb{R}^2 \to \mathbb{R}^2$;

e) $f(x, y) = (\frac{x}{y}, \frac{y}{x}), \quad f : (\mathbb{R}^+)^2 \to (\mathbb{R}^+)^2$;

f) $f(m, n) = (\frac{m}{n}, \frac{n}{m}), \quad f : \mathbb{N}^2 \to \mathbb{R}^2$;

g) $f(m, n) = (m + 2n, n - 2m), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^2$;

h) $f(m, n) = (m - n, n^2, m + n^2), \quad f : \mathbb{Z}^2 \to \mathbb{Z}^3$;

i) $f(m,n) = (m-n, m^2, m^2+m)$, $\quad f : \mathbb{Z}^2 \to \mathbb{Z}^3$;

j) $f(m,n) = (m-n, m+n, m \cdot n)$, $\quad f : \mathbb{Z}^2 \to \mathbb{Z}^3$;

k) $f(x,y,z) = (2x-z, y^2, x+z)$, $\quad f : \mathbb{R}^3 \to \mathbb{R}^3$;

l) $f(x,y,z) = (2x-z, y^3, x+z)$, $\quad f : \mathbb{R}^3 \to \mathbb{R}^3$;

m) $f(m,n,p) = (m+n, n+p, m+p)$, $\quad f : \mathbb{Z}^3 \to \mathbb{Z}^3$;

n) $f(m,n,p) = (m-n, n-p, m-p)$, $\quad f : \mathbb{Z}^3 \to \mathbb{Z}^3$;

o) $f(m,n,p) = (m+2n, n-p, n+p)$, $\quad f : \mathbb{Z}^3 \to \mathbb{Z}^3$;

p) $f(x,y,z) = (\sqrt{x \cdot y}, \sqrt{x \cdot z}, \sqrt{y \cdot z})$, $\quad f : (\mathbb{R}^+)^3 \to (\mathbb{R}^+)^3$;

q) $f(x,y,z) = (x+y+z, x+z)$, $\quad f : \mathbb{R}^3 \to \mathbb{R}^2$.

Exercise 4.8 (*)**.** Denote by $\mathcal{A}^*$ the set of all finite words over alphabet $\mathcal{A}$, including empty word ε, and by $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. Furthermore denote by $\mathcal{A}^n$ the set of all words whose length is exactly n ($n \geq 0$). Decide whether the following mappings are surjective and/or injective. Prove or provide a counterexample.

Note that we strictly distinguish multiplication ($a \cdot b$, always with a dot) from string concatenation (ab, always without a dot).

a) $f(a_1 a_2 \cdots a_m) = m \quad f : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{N}_0$.

b) $f(a_1 a_2 \cdots a_m) = a_m a_{m-1} \cdots a_1$, $\quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$.

c) $f(a_1 a_2 \cdots a_m) = (\mathtt{1}-a_1)(\mathtt{1}-a_2)\cdots(\mathtt{1}-a_m)$, $\quad f : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$.

d) $f(a_1 a_2 \cdots a_n) = a_1 + a_2 + \cdots + a_n$, $\quad f : \{\mathtt{0},\mathtt{1}\}^n \to \mathbb{N}_0,\ n > 1$.

e) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n$, $\quad f : \{\mathtt{0},\mathtt{1}\}^n \to \mathbb{N}_0,\ n \geq 1$.

f) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n$, $\quad f : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{N}_0$.

g) $f(a_1 a_2 \cdots a_n) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{n-1} \cdot a_n$, $\quad f : \{\mathtt{0},\mathtt{1},\mathtt{2}\}^n \to \mathbb{N}_0,\ n > 1$.

h) $f(a_1 a_2 \cdots a_m) = a_1 + 2 \cdot a_2 + 2^2 \cdot a_3 + \cdots + 2^{m-1} \cdot a_m$, $\quad f : \{\mathtt{0},\mathtt{1},\mathtt{2}\}^* \to \mathbb{N}_0$.

i) $f(a_1 a_2 \cdots a_m) = a_2 a_3 \cdots a_m a_1$, $\quad f : \{\mathtt{0},\mathtt{1}\}^+ \to \{\mathtt{0},\mathtt{1}\}^+$.

j) $f(a_1 a_2 \cdots a_m) = a_2 a_3 \cdots a_m \mathtt{0}$, $\quad f : \{\mathtt{0},\mathtt{1}\}^+ \to \{\mathtt{0},\mathtt{1}\}^+$.

k) $f(a_1 a_2 \cdots a_m) = a_1(a_1 \cdot a_2)(a_2 \cdot a_3)\cdots(a_{m-1} \cdot a_m)$, $\quad f : \{\mathtt{0},\mathtt{1}\}^+ \to \{\mathtt{0},\mathtt{1}\}^+$.

4.3 Cardinality of sets

Exercise 4.9. Let us consider the three sets

$$A = \{1,2,3\}, \qquad B = \{3,4,5,6\} \qquad \text{and} \qquad C = \{2,4,6\}.$$

Compute the following cardinalities: $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, $|A \cup B|$, $|A \cup C|$, $|B \cup C|$, $|A \cap B \cap C|$, $|A \cup B \cup C|$, $|A \times B|$, $|A \times C|$, $|B \times C|$, $|A \times B \times C|$, $|\mathcal{P}(A)|$.

Exercise 4.10. Let A, B, C be three sets with cardinality respectively $|A| = 3$, $|B| = 4$, $|C| = 5$. Suppose that $|A \cap B| = 1$, $|B \cap C| = 2$ and $|A \cap C| = 3$ and that only one element belongs to all three sets at the same time. What is the cardinality of the set $A \cup B \cup C$?

Exercise 4.11. Let A, B, C be three finite sets such that

$$|B| = 4,\ |C| = 5,\ |A \cap B| = 3,\ |A \cap C| = 2,\ |B \cap C| = 2,\ |A \cap B \cap C| = 2 \quad \text{and} \quad |A \cup B \cup C| = 9.$$

What is the cardinality of A?

Exercise 4.12. Let A, B, C be three (finite or infinite) sets. Prove the following properties.

i) $|A| = |A|$ and $|A| \leq |A|$;

ii) if $A \subseteq B$, then $|A| \leq |B|$;

iii) if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$;

iv) if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$;

v) (*) if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Exercise 4.13. Let $2\mathbb{Z}$ be the set of even numbers. Prove that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. What is the cardinality of $\mathbb{Z} \times 2\mathbb{Z}$? And the cardinality of $\{\pi\} \times 2\mathbb{Z}$?