

Mathematics for Informatics

Constrained optimization, multivariate integration
(lecture 7 of 12)

Francesco DOLCE

dolcefra@cvut.cz

Czech Technical University in Prague

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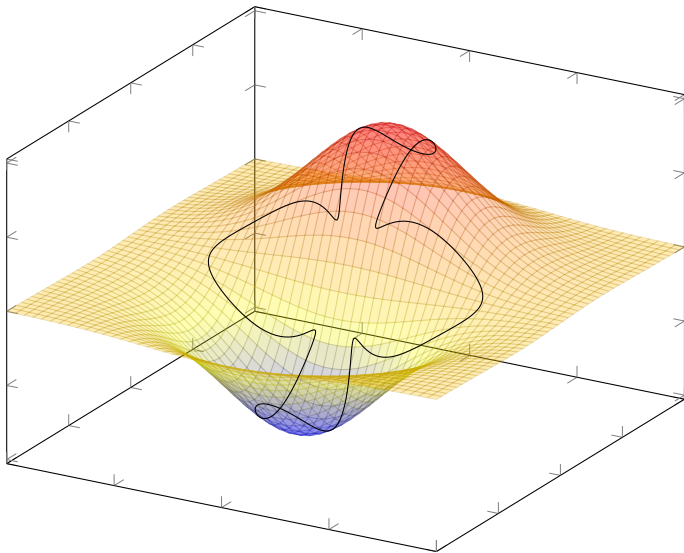
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Outline

- Constrained optimization
- Reminder: integration of functions of 1 variable
- 2-variate function integration

Motivation

Find the maximum and minimum points when walking along the black line:



The problem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

Find (local) maxima and minima of f subject to

$$\begin{cases} g_1(x_1, x_2, \dots, x_n) &= 0 \\ g_2(x_1, x_2, \dots, x_n) &= 0 \\ &\vdots \\ g_p(x_1, x_2, \dots, x_n) &= 0. \end{cases}$$

Set $\mathcal{G} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid g_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, p\}$.

Assumptions

- 1 The functions f and g_i , with $i = 1, 2, \dots, p$, have continuous second partial derivatives.

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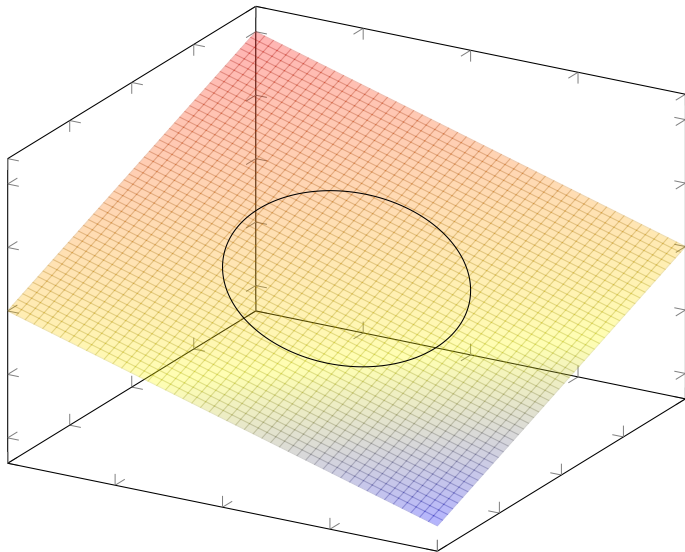
Example

Are the gradients of the following functions linearly independent?

$$\begin{aligned} g_1(x, y) &= 2x + xy^2, & g_2(x, y) &= 4x + 2xy^2, \\ g_3(x, y) &= 2xy^2 + 4y^2, & g_4(x, y) &= 2x + 3xy^2 + 4y^2. \end{aligned}$$

Running example

$$f(x, y) = x + y, \quad \mathcal{G} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}$$



Necessary condition

Theorem

Assume f has a local extremum in $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{G}$ subject to $\mathcal{G}$.

Then there exist numbers $\mu_1^*, \dots, \mu_p^*$ such that the **Lagrangian function** L given by

$$L(x_1, \dots, x_n, \mu_1, \dots, \mu_p) = f(x_1, \dots, x_n) + \sum_{i=1}^p \mu_i g_i(x_1, \dots, x_n)$$

has zero partial derivatives with respect to $x_1, \dots, x_n$ at the point x^* .

In other words, the following system is true:

$$\left\{ \begin{array}{rcl} \frac{\partial f}{\partial x_1}(x^*) + \mu_1^* \frac{\partial g_1}{\partial x_1}(x^*) + \dots + \mu_p^* \frac{\partial g_p}{\partial x_1}(x^*) & = & 0 \\ & \vdots & \\ \frac{\partial f}{\partial x_n}(x^*) + \mu_1^* \frac{\partial g_1}{\partial x_n}(x^*) + \dots + \mu_p^* \frac{\partial g_p}{\partial x_n}(x^*) & = & 0 \end{array} \right.$$

Sufficient condition

Theorem

Let $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ and $\mu^* = (\mu_1^*, \dots, \mu_p^*) \in \mathbb{R}^p$ such that

- (i) the Lagrangian function $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$ has zero partial derivatives with respect to $x_1, \dots, x_n$ at the point $(x^*, \mu^*) \in \mathbb{R}^{n+p}$;
- (ii) the Lagrangian function $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$ has zero partial derivatives with respect to $\mu_1, \dots, \mu_p$ at the point $(x^*, \mu^*) \in \mathbb{R}^{n+p}$;
- (iii) for all non-zero $y \in \mathbb{R}^n$ satisfying $y \cdot \nabla g_i(x^*) = 0$ for $i = 1, 2, \dots, p$ we have

$$y \left(\nabla^2 f(x^*) + \sum_{i=1}^p \mu_i^* \nabla^2 g_i(x^*) \right) y^T > 0.$$

Thus, the function f has a strict local minimum at x^* (subject to $\mathcal{G}$).

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- ❸ for all non-zero $y \in \mathbb{R}^n$ satisfying $y \cdot \nabla g_i(x^*) = 0$ for $i = 1, 2, \dots, p$ we have

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Thus, the function f has a strict local minimum at x^* (subject to $\mathcal{G}$).

If we replace in (iii) the condition “ > 0 ” by “ < 0 ”, we obtain a sufficient condition of a strict local maximum.

Example

Example

Find maxima and minima of $f(x, y) = x + y$ subject to $x^2 + y^2 = 2$.

Integration of functions of 1 variable

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a < b$.

Recall what does $\int_a^b f(x) dx$ mean, if it exists.

What is its geometrical meaning?

$\int_a^b f(x) dx$ measures the **signed** area accumulated by f between a and b .

Lebesgue integral - measurable sets

Contrary to Riemann's integral, instead of cutting the interval into "thin" vertical pieces, we group points where f has the same value.

This allows us to integrate functions with many discontinuities (provided the relevant sets can be measured).

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A set A is **measurable** if we can assign a well-defined length $\mu(A)$ (e.g., $\mu(A) = |A|$).

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A function f is **measurable** in A if the set $\{x \in A : f(x) > t\}$ is measurable for every $t \in \mathbb{R}$.

Lebesgue integral - simple functions

A non-negative function s is **simple** if takes only finitely many values

$$s(x) = \sum_{i=1}^N c_i \cdot \chi_{A_i}(x)$$

for $c_i \geq 0$ and $A_1, A_2, \dots, A_N$ pairwise disjoint measurable sets.

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In this case, if $[a, b] = \bigsqcup_{i=1}^n A_i$, we define

$$\int_a^b s(x) \, dx = \sum_{i=1}^N c_i \cdot \mu(A_i).$$

Lebesgue integral - definition

For a measurable function $f: [a, b] \rightarrow [0, +\infty]$ we define

$$\int_a^b f(x) \, dx = \sup \left\{ \int_a^b s(x) \, dx : 0 \leq s \leq f, s \text{ simple} \right\}.$$

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If $f: [a, b] \rightarrow \mathbb{R}$,

$$\int_a^b f(x) \, dx = \int_a^b f^+(x) \, dx - \int_a^b f^-(x) \, dx$$

(if both terms are finite), where $f(x) = f^+(x) - f^-(x)$, with

$$f^+(x) = \max(f(x), 0) \quad \text{and} \quad f^-(x) = \max(0, -f(x)).$$

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If $\int_a^b f(x) \, dx < \infty$, we say that f is **Lebesgue integrable**.

Lebesgue integral - A few properties

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If f is integrable on $[a, b]$ and on $[b, c]$, then it is integrable on $[a, c]$ and

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.$$

Primitive function

Let $F(x)$ be a real function, continuous on $[a, b]$ and differentiable on (a, b) .

Let $f(x)$ be a real function, continuous on (a, b) and such that

$$\forall x \in (a, b), \quad F'(x) = f(x).$$

Such function F is called a primitive function of f on (a, b) .

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Such function F is called a **primitive function of f on (a, b)** .

Example

Find a primitive function on $(0, 1)$ of the function $f(x) = 2x + x^2$.

Fundamental Theorem of Calculus

Let f be integrable on $[a, b]$ and $F(x)$ be (one of) its primitive function on (a, b) .
We have

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a).$$

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Example

Calculate $\int_0^1 (2x + x^2) dx$.

Substitution

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.

Let φ be a real function differentiable on (α, β) such that φ and φ' are both continuous on $[\alpha, \beta]$.

Let f be continuous on $[\varphi(\alpha), \varphi(\beta)]$ (if $\varphi(\alpha) \leq \varphi(\beta)$, otherwise continuous on $[\varphi(\beta), \varphi(\alpha)]$).

If $f(\varphi(t)) \cdot \varphi'(t)$ is integrable on $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot |\varphi'(t)| \, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) \, dx.$$

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Example

Calculate $\int_1^2 \frac{2 \ln(t)^2}{t} dt$.

Integration by parts

Let f and g be differentiable on (a, b) and let f, g, f', g' be continuous on $[a, b]$. We have

$$\int_a^b f'(x) \cdot g(x) \, dx = [f(x) \cdot g(x)]_a^b - \int_a^b f(x) \cdot g'(x) \, dx.$$

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Example

Calculate $\int_1^2 10x \ln x \, dx$.

2-variate function

Suppose we have a function $f : D \rightarrow [0, +\infty]$, where $D \subset \mathbb{R}^2$ is measurable.

Imagine that this function represents (part of) a surface of some object.

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Imagine that this function represents (part of) a surface of some object.

The point $z = f(x, y)$ describes the height above the point (x, y) .

The integral $\int_D f(x, y) \, d(x, y)$ is the volume under the surface.

2-variate simple functions

A non-negative simple function s on a measurable set $D \subset \mathbb{R}^2$ has the form

$$s(x, y) = \sum_{i=1}^N c_i \cdot \chi_{A_i}(x, y)$$

for $c_i \geq 0$ and $A_i \subset D$.

In this case,

$$\int_D s(x, y) \, d(x, y) = \sum_{i=1}^N c_i \cdot \mu(A_i),$$

with, e.g., $\mu(A)$ the area of A .

Lebesgue integral in 2 variables

For a measurable function $f: D \rightarrow [0, +\infty]$ we define

$$\int_D f(x, y) \, d(x, y) = \sup \left\{ \int_D s(x, y) \, d\mu : 0 \leq s \leq f, s \text{ simple} \right\}.$$

If $f: D \rightarrow \mathbb{R}$, we use $f = f^+ - f^-$ as in dimension 1, and say that f is **Lebesgue integrable** if

$$\int_D |f(x, y)| \, d(x, y) < \infty.$$

Fubini's Theorem

Theorem

If f is integrable over $D = [a, b] \times [c, d]$, then

$$\int_D f(x, y) \, d(x, y) = \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy.$$

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Example

Calculate the double integral over $D = [0, 2] \times [-1, 2]$ of the function $f(x, y) = x^2y + 1$.

And if D is not a rectangle?

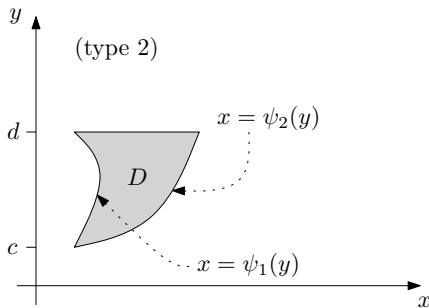
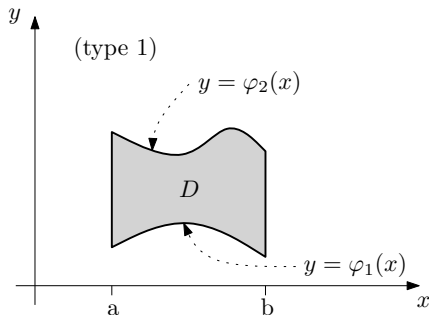
If D is measurable, we can extend the function by zero outside of D :

$$\int_D f(x, y) \, d(x, y) = \int_{R^2} f(x, y) \cdot \chi_D(x, y) \, d(x, y).$$

Special types of domain D (1/2)

We will consider the following two types of the domain D .

- (type 1) $x \in [a, b]$ and y is bounded by continuous functions $\varphi_1(x)$ and $\varphi_2(x)$;
- (type 2) $y \in [c, d]$ and x is bounded by continuous functions $\psi_1(y)$ and $\psi_2(y)$.



Special types of domain D (2/2)

Double integrals over such D are calculated as follows.

Theorem

If the integral on the right side exists, then we have (for such a domain D):

- if D is of **type 1**, then

$$\iint_D f(x, y) \, d(x, y) = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, dy \right) dx;$$

- if D is of **type 2**, then

$$\iint_D f(x, y) \, d(x, y) = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) \, dx \right) dy.$$

Example

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Let D be the region given by the triangle with vertices $(0, 1)$, $(0, 2)$ and $(3, 0)$. Calculate

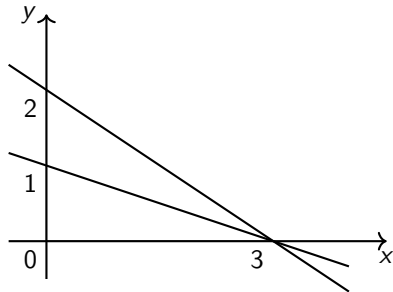
$$\iint_D \frac{x+y}{2} \, d(x, y).$$

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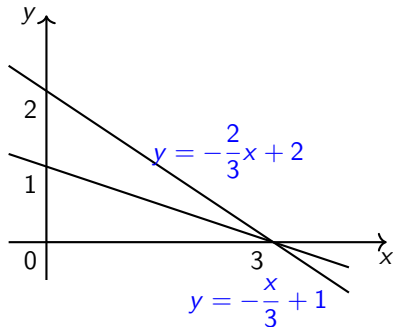


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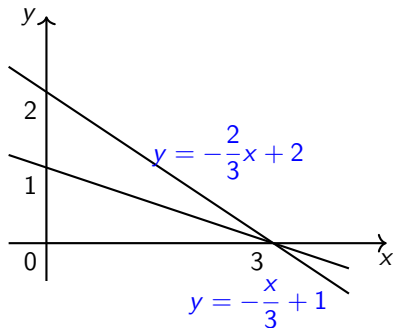


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$$\int_0^3 \int_{1-\frac{x}{3}}^{2-\frac{2}{3}x} \frac{x+y}{2} dy dx = \dots = \frac{3}{2}$$