
REVIEW - RANDOM VECTORS I: INDEPENDENCE, CONDITIONALS

Joint distribution:

If X, Y are discrete random variables with values x_i, y_j respectively, their joint distribution is given by the probabilities $P(X = x_i \cap Y = y_j)$ for all possible (x_i, y_j) .

If (X, Y) is continuous with a density $f(x, y)$, then the joint distribution is given by the joint density $f_{XY}(x, y)$.

Marginal distribution:

If (X, Y) has a discrete distribution with values (x_i, y_j) , then the marginal distribution of X is discrete and can be computed as

$$P(X = x_i) = \sum_j P(X = x_i \cap Y = y_j).$$

If (X, Y) is continuous with a density $f(x, y)$, then the marginal density of X is computed as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Similarly for the marginal distribution of Y .

Independence:

Two random variables X and Y are independent if

$$P(X \leq x \cap Y \leq y) = P(X \leq x) \cdot P(Y \leq y) \quad \forall (x, y) \in \mathbb{R}^2.$$

If X, Y are continuous random variables, then X and Y are independent if and only if $f_{XY}(x, y) = f_X(x)f_Y(y)$ for almost all $(x, y) \in \mathbb{R}^2$.

If X, Y are discrete random variables, then X and Y are independent if and only if $P(X = x_i \cap Y = y_j) = P(X = x_i)P(Y = y_j)$ for all possible (x_i, y_j) .

Conditional distribution:

- a) If X, Y are discrete random variables with values x_i, y_j respectively,, then the conditional distribution of X given $Y = y$ is discrete and can be computed as (provided $P(Y = y_j) > 0$)

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}.$$

- b) If X, Y are continuous random variables with joint density $f_{X,Y}(x, y)$, then the conditional density of X given $Y = y$ is computed as (provided $f_Y(y) > 0$)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Conditional expectation:

If (X, Y) has a discrete distribution with values (x_i, y_j) , then the conditional expectation of X given $Y = y$ is defined as

$$E(X|Y = y_j) = \sum_{x_i} x_i P(X = x_i | Y = y_j).$$

If (X, Y) has a continuous distribution, then the conditional expectation of X given $Y = y$ is defined as

$$E(X|Y = y) = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y) dx.$$

EXERCISES 6 - RANDOM VECTORS I: INDEPENDENCE, CONDITIONALS

1. Suppose that a random vector (X, Y) has the following joint probability distribution:

$P(X = x \cap Y = y)$		Y		
		0	1	2
X	1	0.4	0.15	0.05
	2	0.3	0.06	0.04

- Find the marginal distributions of the random variables X and Y .
- Are X and Y independent?
- Find the conditional distribution of $X|Y = y$.
- Find $E(X|Y = y)$.
- Find the conditional distribution of $X + Y$ given $Y = y$.

2. Consider two continuous random variables X and Y with joint density

$$f(x, y) = \begin{cases} \frac{1}{5}(4xy + 4x - 8y) & \text{for } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Are X and Y independent?
- Find the conditional density $f_{Y|X}(y|x)$.
- Find $E(Y|X = x)$.
- Find $E(X + Y)$. (*See next Tutorial*)

3. Consider two continuous random variables X and Y with the joint density

$$f(x, y) = \begin{cases} \frac{1}{2}ye^{(-x-y^2)/2} & \text{for } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Are X and Y independent?
- Find $E(X + Y)$. (*See next Tutorial*)
- Find $E(XY)$. (*See next Tutorial*)

4. Let X and Y be two continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{1}{3}(-4xy - 4x + 8y + 8) & \text{for } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Are X and Y independent?
- Find $E(X + Y)$. (*See next Tutorial*)
- Find $E(XY)$. (*See next Tutorial*)

5. Let X and Y be continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{3}{8}xy^2 & \text{for } 0 \leq x \leq 1, \text{ and } -2 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

- Find the marginal density of the random variable X .
- Find the conditional density $f_{Y|X}(y|x)$.

6. Suppose we roll two dice. Let X be the random variable defined as the sum of the two numbers rolled, and Y the random variable is defined as the difference between the number rolled on the first and the number rolled on the second die. (i.e., $Y = 1^{st} - 2^{nd}$). Find $E(X|Y = 2)$.

ADDITIONAL EXERCISES - RANDOM VECTORS I: INDEPENDENCE, CONDITIONALS

7. Two people are betting on the outcome of a single roll of a (fair) six-sided die. If a person is correct in their bet, they win 1. If the person is not correct, they lose 1. Suppose that person X bets on even numbers and person Y bets on large numbers $\{4, 5, 6\}$. Let X denote a random variable marking the winnings/losings of X after one game, same for Y .

- a) Find the joint distribution of X and Y .
- b) Find the marginal distributions for X and Y .
- c) Are X and Y independent?
- d) Find the conditional distribution of X given $Y = 1$ and given $Y = -1$.
- e) Find the covariance and the coefficient of correlation between X and Y . (*See next Tutorial*)