

Elementary Introduction to Graph Theory

(01YEIG 2026/2027)

Lecture 2



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PDF available at the address:
dolcefra.pages.fit/ens/2627/YEIG-lecture-02.pdf

Solutions of Exercises in the previous Lecture

Solution of Exercise [Degree of $K_{n,m}$].

$$\delta(K_{m,n}) = \min\{m, n\}, \quad \Delta(K_{m,n}) = \max\{m, n\}.$$

1 Graphic sequences (continued)

Recall that a sequence $(d_i)_{i=1}^n$ is called *graphic* if there exists a simple graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ such that for every $1 \leq i \leq n$ we have $d_G(v_i) = d_i$. In this case we say that G *realises* the sequence $(d_i)_{i=1}^n$.

Let us start by proving the first result characterising graphic sequences.

Theorem 1 (Havel (1955), Hakimi (1961)) *A non-increasing non-negative sequence $(d_i)_{i=1}^n$ is graphic if and only if the sequence obtained by subtracting 1*

from the next d_1 entries and then reordering into non-increasing order is graphic, i.e., if and only if

$$(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \dots, d_n)$$

is graphic.

To prove Havel-Hakimi Theorem we need the following result.

Lemma 1 Let $(d_i)_{i=1}^n$ be a graphic sequence with $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$. Then, there is a simple graph $G = (V, E)$ realising $(d_i)_{i=1}^n$ such that $V = \{v_1, \dots, v_n\}$, $d_G(v_i) = d_i$ for every i , and $N_G(v_1) = \{v_2, \dots, v_{d_1+1}\}$.

Proof. Let $S = \{v_2, \dots, v_{d_1+1}\}$. From all graphs realising $(d_i)_{i=1}^n$ let us select G s.t. $r = |N_G(v_1) \cap S|$ is maximum.

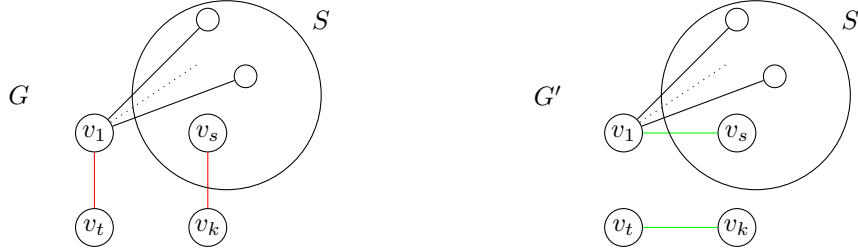
If $r = d_1$, then G is the graph we are looking for and we are done.

If, by contradiction, $r < d_1$, then there exist s, t with $2 \leq s \leq d_1 + 1 < t \leq n$ such that $\{v_1, v_t\} \in E$ and $\{v_1, v_s\} \notin E$. Since $d_G(v_s) \geq d_G(v_t)$ and $\{v_1, v_t\} \in E$, there exists a vertex v_k such that

$$\{v_s, v_k\} \in E \quad \text{and} \quad \{v_t, v_k\} \notin E.$$

Indeed, otherwise $N_G(v_s) \subsetneq N_G(v_t)$, since $v_1 \in N_G(v_t) \setminus N_G(v_s)$, implying $d_G(v_s) < d_G(v_t)$, a contradiction.

Let $G' = (V, E')$, with $E' = (E \setminus \{\{v_1, v_t\}, \{v_s, v_k\}\}) \cup \{\{v_1, v_s\}, \{v_t, v_k\}\}$.



We have $|N_{G'}(v_1) \cap S| = r + 1 > r$, which contradicts the maximality of r . ■

Proof of Havel-Hakimi Theorem.

($\Rightarrow$) If $(d_i)_{i=1}^n$ is graphic, by Lemma 1 there exists a graph G realising it with $N_G(v_1) = \{v_2, \dots, v_{d_1+1}\}$. Thus, the graph obtained from G by removing the vertex v_1 and all its adjacent edges (such graph will be denoted $G - \{v_1\}$) has degree sequence $(d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$.

($\Leftarrow$) If $(d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic, there exists a graph $G' = (V', E')$ with $\#V' = n - 1$, $V' = \{v_2, \dots, v_n\}$ such that

$$d_{G'}(v_i) = \begin{cases} d_i - 1 & \text{for } 2 \leq i \leq d_{d_1+1} \\ d_i & \text{for } d_{d_1+2} \leq i \leq n \end{cases}.$$

Thus the graph $G = \left(V' \cup \{v_1\}, E' \cup \bigcup_{i=2}^{d_1+1} \{v_1, v_i\} \right)$ realises $(d_i)_{i=1}^n$. ■

Havel-Hakimi Theorem gives us a recursive algorithm to check whether a sequence is graphic. This algorithm can be implemented in polynomial time.

Algorithm 1: $\text{GraphicSequence}(d_1, d_2, \dots, d_n)$

Input: non-increasing sequence $(d_1, d_2, \dots, d_n)$

Output: TRUE if $(d_i)_{i=1}^n$ is graphic; FALSE if not

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1 if  $n = 0$  then
2   return TRUE
3 else if  $(d_1 > n - 1)$  or  $(d_n < 0)$  then
4   return FALSE
5 else if  $d_1 = 0$  then
6   return TRUE
7 else
8   Let  $(a_1, a_2, \dots, a_{n-1})$  be the non-increasing rearrangement of
       $(d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ 
9   return  $\text{GraphicSequence}(a_1, a_2, \dots, a_{n-1})$ 

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Example 1

- $(2, 1)$ is not a graphic sequence, since $2 > 1$.
- $(\mathbf{3}, \underline{2}, 1, 1) \rightarrow (\mathbf{1}, \underline{0}, 0) \rightarrow (-1, 0) \sim (0, -1)$ is not graphic since $-1 < 0$.
- $(\mathbf{2}, 2, \underline{1}, 1) \rightarrow (1, 0, 1) \sim (\mathbf{1}, \underline{1}, 0) \rightarrow (0, 0)$ is graphic (see Figure 1).

We can also use the algorithm backwards to realise a graphic sequence: we start with as many vertices as there are 0s in the last sequence and then, at each step, add a vertex with edges according to its degree (see Figure 1).

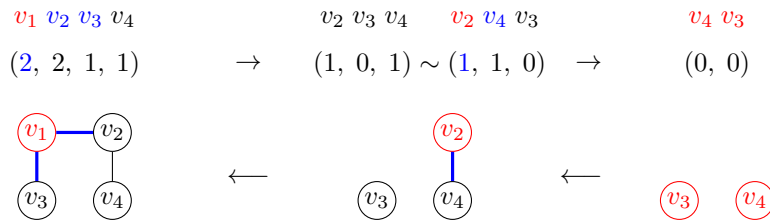


Figure 1: Realising a graph from a graphic sequence.

Exercise [Havel-Hakimi Theorem]. Determine, using the Havel-Hakimi Theorem, which of the following sequences are graphic, and in the positive case find a graph realising them: $(7, 6, 5, 4, 3, 3, 2)$, $(3, 3, 2, 2, 1, 1)$.

We also have the following non-recursive result.

Theorem 2 (Erdős, Gallai (1960)) *A sequence $(d_i)_{i=1}^n$ with $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ is graphic if and only if*

$$\sum_{i=1}^n d_i \text{ is even} \quad \text{and} \quad \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\} \quad \forall 1 \leq k \leq n.$$

While the Havel-Hakimi Theorem is constructive, with the Erdős-Gallai Theorem we have a direct characterisation.

Example 2 The sequence $(3, 3, 1, 1)$ is not graphic. Indeed, $3 + 3 + 1 + 1$ is even, and

$$d_1 = 3 \leq 3 = 1(1-0) + \min\{1, 3\} + \min\{1, 1\} + \min\{1, 1\}$$

but

$$d_1 + d_2 = 6 > 4 = 2(2-1) + \min\{2, 1\} + \min\{2, 1\}.$$

Exercise [Erdős-Gallai Theorem]. Determine, using the Erdős-Gallai Theorem, whether the sequence $(4, 4, 2, 2, 2)$ is graphic.

2 Graph operations

Given two (simple) graphs $G = (V, E)$ and $H = (U, F)$ we can combine them in several ways to obtain new graphs.

Their *union* is defined as $G \cup H = (V \cup U, E \cup F)$. Note that if $V \cap U = \emptyset$ we just obtain a "juxtaposition" of the two graphs.

Their *intersection* is defined as $G \cap H = (V \cap U, E \cap F)$.

Example 3 Let us consider $G = (\{a, b, c, d\}, \{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}\})$ and $H = (\{a, b, c, e\}, \{\{a, b\}, \{a, c\}, \{b, e\}, \{c, e\}\})$ (see Figure 2). Then

$$G \cup H = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, e\}\})$$

and

$$G \cap H = (\{a, b, c\}, \{\{a, b\}\}).$$

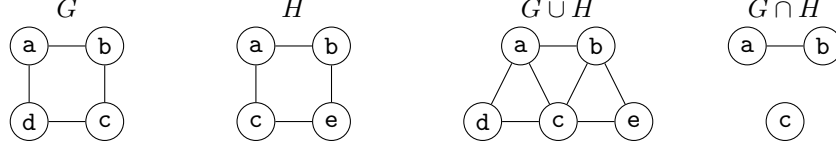


Figure 2: Union and intersection of two graphs.

The *complement* of a simple graph $G = (V, E)$ is the graph

$$\bar{G} = \left(V, \binom{V}{2} \setminus E \right).$$

Example 4 Let us consider $G = (\{a, b, c, d\}, \{\{a, b\}, \{a, c\}, \{b, c\}\})$. Then, $\bar{G} = (\{a, b, c, d\}, \{\{a, d\}, \{b, d\}, \{c, d\}\})$ (see Figure 3).

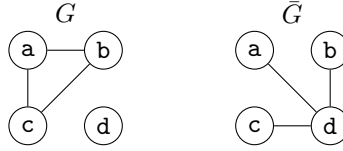


Figure 3: A graph and its complement.

If two graphs $G = (V, E_1)$ and $H = (V, E_2)$ are defined over the same set of vertices, we can define their *symmetric difference* $G \triangle H$ as

$$G \triangle H = (V, E_1 \triangle E_2), \quad \text{where} \quad E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1).$$

Example 5 Let $G = (V, E_1)$ and $H = (V, E_2)$, where $V = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}\}$, and $E_2 = \{\{a, b\}, \{b, d\}, \{c, d\}\}$. Then $G \triangle H = (V, \{\{a, d\}, \{b, d\}, \{b, c\}\})$ (see Figure 4).

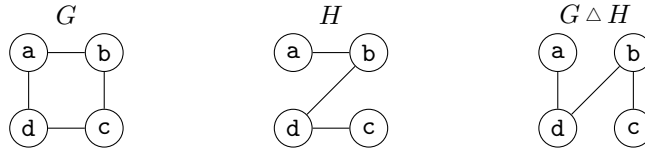


Figure 4: Symmetric difference of two graphs.

The *Cartesian product* $G \square H$ of the two graphs is defined as the graph having set of vertices $V \times U$ and edges exactly the pairs $\{(a, u), (b, v)\}$ such that

$$(\{a, b\} \in E \text{ and } u = v) \quad \text{or} \quad (a = b \text{ and } \{u, v\} \in F).$$

Example 6 Let us consider the graphs $G = (\{a, b, c\}, \{\{a, b\}, \{a, c\}, \{b, c\}\})$ and $H = (\{1, 2\}, \{\{1, 2\}\})$. The Cartesian product $G \square H$ is represented in Figure 5.

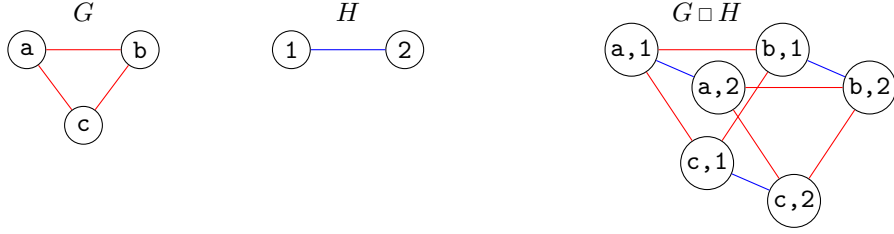


Figure 5: Cartesian product of two graphs.

3 Subgraphs

Let $G = (V, E)$ be a graph. A graph $H = (U, F)$ is a *subgraph* of G , written $H \subseteq G$ (or, G is a *supergraph* of H , written $G \supseteq H$) if $U \subseteq V$ and $F \subseteq E \cap \binom{U}{2}$. We say that H is *contained in* G (or, G *contains* H).

Example 7 The graphs H_1, H_2, H_3 in Figure 6 are subgraphs of G . On the other hand, H_4 is not a subgraph of G .

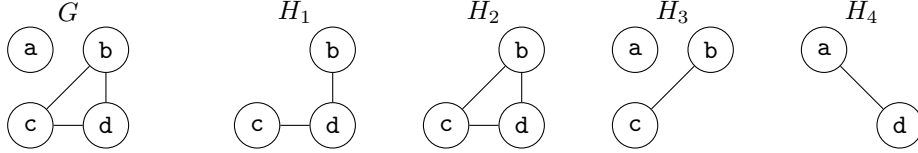


Figure 6: Five graphs with vertices (subsets of) $\{a, b, c, d\}$.

A subgraph $H = (U, F)$ of $G = (V, E)$ is called an *induced subgraph* of G by U , and we write $H = G[U]$, if for every two vertices $x, y \in U$ we have $\{x, y\} \in F \Leftrightarrow \{x, y\} \in E$. That is, the vertices in H are connected by exactly the same edges as in G . When H is simple we have $H = G[U] = \left(U, E \cap \binom{U}{2}\right)$.

Example 8 Let G, H_1, H_2 and H_3 be the graphs seen in Example 7. We have $H_2 = G[\{b, c, d\}]$ and $H_3 = G[\{a, b, c\}]$. Note that H_1 is not an induced subgraph of G , since $\{b, c\}$ is an edge of G but not of H_1 .

If $U \subset V$ and $F \subset \binom{V}{2}$, we write $G - U = G[V \setminus U]$. That is, $G - U$ is obtained from G by deleting all vertices in U and their incident edges.

Given a graph $G = (V, E)$ and a set of edges F , the graph $G - F = (V, E \setminus F)$ is obtained by deleting some edges, the ones in $E \cap F$, but keeping all vertices. We call $G - F$ a *spanning subgraph* of G .

Similarly, given a set of vertices U and a set of edges F (whose endpoints are in V), we define $G + U = (V \cup U, E)$ and $G + F = (V, E \cup F)$.

When $U = \{u\}$ we just write $G + u$ instead of $G + \{u\}$ and $G - u$ instead of $G - \{u\}$. Similarly, when $F = \{e\}$ we write $G - e$ and $G \setminus e$.

Example 9 Let G, H_1, H_2, H_3 and H_4 be as in Example 7. Then $H_2 = G - \mathbf{a}$ and $H_3 = G - \mathbf{d}$. The graph H_1 is a spanning subgraph of H_2 , indeed $H_1 = H_2 - \{\mathbf{b}, \mathbf{c}\}$. Obviously we also have $H_2 = H_1 + \{\mathbf{b}, \mathbf{c}\}$. Moreover $G = H_2 + \mathbf{a}$.

4 Paths and cycles

A *path* is a non-empty graph $P = (V, E)$ of order $k + 1$ with $V = \{v_0, v_1, \dots, v_k\}$ and $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}\}$. The vertices v_0, v_k are *linked* by P and are called its *endpoints*. The vertices $v_1, v_2, \dots, v_{k-1}$ are the *inner vertices* of P .