

Elementary Introduction to Graph Theory

(01YEIG 2026/2027)

Lecture 3



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dolcefra.pages.fit/ens/2627/YEIG-lecture-03.pdf

Solutions of Exercises in the previous Lecture

Solution of Exercise [Havel-Hakimi Theorem].

- $(7, 6, 5, 4, 3, 3, 2)$ is not a graphic sequence, since $7 > 6$.
- $(\mathbf{3}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \rightarrow (0, 0, 1, 1) \sim (\mathbf{1}, \mathbf{1}, 0, 0) \rightarrow (0, 0, 0)$ is graphic (see Figure 1).

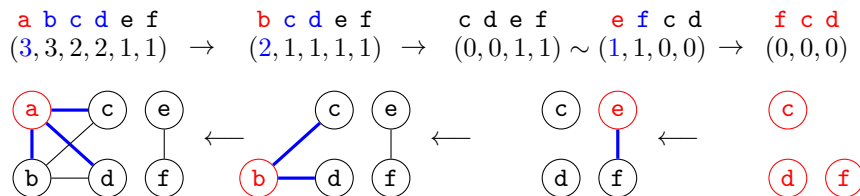


Figure 1: Realising a graph from a graphic sequence.

Solution of Exercise [Erdős-Gallai Theorem]. The sequence $(d_i)_{i=1}^5 = (4, 4, 2, 2, 2)$ is graphic. Indeed, the sum $\sum_{i=1}^5 d_i = 4 + 4 + 2 + 2 + 2 = 14$ is even, and

$$\begin{aligned} 4 &\leq 4 = 1(1-0) + \min\{1, 4\} + \min\{1, 2\} + \min\{1, 2\} + \min\{1, 2\} \\ 4 + 4 &= 8 \leq 8 = 2(2-1) + \min\{2, 2\} + \min\{2, 2\} + \min\{2, 2\} \\ 4 + 4 + 2 &= 10 \leq 10 = 3(3-1) + \min\{3, 2\} + \min\{3, 2\} \\ 4 + 4 + 2 + 2 &= 12 \leq 14 = 4(4-1) + \min\{4, 2\} \\ 4 + 4 + 2 + 2 + 2 &= 14 \leq 20 = 5(5-1) \end{aligned}$$

1 Paths and cycles (continued)

A *path* is a non-empty graph $P = (V, E)$ of order $k+1$ with $V = \{v_0, v_1, \dots, v_k\}$ and $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}\}$. The vertices v_0, v_k are *linked* by P and are called its *endpoints*. The vertices $v_1, v_2, \dots, v_{k-1}$ are the *inner vertices* of P .

If $k \geq 3$, we call *cycle* a graph of order k of the form $C = (V, E)$ with $V = \{v_0, v_1, \dots, v_{k-1}\}$ and $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-2}, v_{k-1}\}, \{v_{k-1}, v_0\}\}$.

The *length* of P (resp., of C) is k . A path (resp., cycle) is *odd* or *even* according to the parity of k . When a path (resp., cycle) appears as a subgraph of G , we say that the path (resp., cycle) is *in* G .

We can represent a path (resp., a cycle) as a sequence of vertices

$$(v_0, v_1, \dots, v_k)$$

or as a sequence of edges

$$(e_1, e_2, \dots, e_k)$$

or, when interested in both vertices and edges, using the notation

$$(v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_{k-1}} v_{k-1} \xrightarrow{e_k} v_k)$$

where $e_i = \{v_{i-1}, v_i\}$.

The *distance* between two vertices $u, v \in V$ in a graph $G = (V, E)$, denoted by $d_G(u, v)$, is the length of the shortest path in G with endpoints u and v . If there is no path connecting u and v , we set $d_G(u, v) = \infty$.

The *eccentricity* of a vertex v is defined as $\text{ecc}_G(v) = \max\{d_G(v, u) \mid u \in V\}$ (this may also be infinite).

The *centre* of G is the set of its vertices with minimal eccentricity, i.e.,

$$\mathcal{C}(G) = \operatorname{argmin}_{v \in V} \text{ecc}_G(v) = \left\{ v \in V \mid \text{ecc}_G(v) = \min_{u \in V} \text{ecc}_G(u) \right\}.$$

Example 1 The path P_4 and the cycle C_3 are represented on the left of Figure 2. On the right of the same figure you can see a **path** of length 4, **(b, c, d, e, f)**, and a **cycle** of length 4, **(b, c, d, e)**, in a graph G .

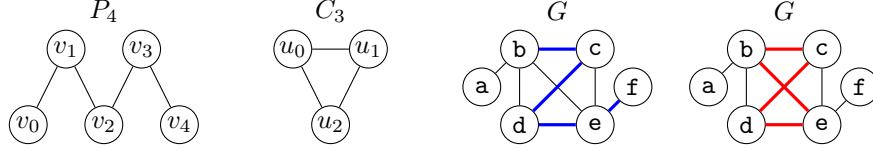


Figure 2: Paths and cycles.

Note that we have $d_G(b, f) = 2$ since a shortest path in G between the two vertices is **(b, e, f)**. One can check that, e.g., $\text{ecc}_{P_4}(v_0) = 4$, $\text{ecc}_{P_4}(v_3) = 3$; $\text{ecc}_{C_3}(u_2) = 1$; $\text{ecc}_G(a) = 3$, and $\text{ecc}_G(c) = 2$. Moreover $\mathcal{C}(P_4) = \{v_2\}$, $\mathcal{C}(C_3) = \{u_0, u_1, u_2\}$, and $\mathcal{C}(G) = \{b, c, d, e\}$.

2 Walks

Given a simple graph $G = (V, E)$, a sequence $(v_i)_{i=0}^k$ of vertices $v_i \in V$ is called a *walk* (or *k-walk*) in G if for every $1 \leq j \leq k$ we have $\{v_{j-1}, v_j\} \in E$.

A walk is *closed* if its initial and terminal vertices are the same. A walk in which all its edges are distinct is called a *trail*.

The notions of *length*, *endpoints*, *inner vertices* in a walk are defined analogously as in a path. Nevertheless, note that, contrary to paths (resp., cycles), vertices in a walk (resp., closed walk) can be repeated.

Example 2 Let G be the graph in Figure 3. The sequence **(d, a, c, a, b)** is a walk in G but not a path. The sequence **(a, c, b, c, a)** is a closed walk in G but not a cycle. The sequence **(d, a, b, c)** is both a walk and a path. The sequence **(d, a, b, c, a)** is a trail (but not a path nor a cycle).

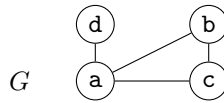


Figure 3: A graph G .

Connectivity by walks is an equivalence relation on V . Indeed:

- every vertex is connected by a 0-walk to itself;
- if there is a walk $W = (v_i)_{i=0}^k$ from $a = v_0$ to $b = v_k$, then the reverse walk $\widetilde{W} = (v_{k-i})_{i=0}^k$ connects b to a ;

- if $a = v_0$ and $b = v_k$ are connected by a walk $W_1 = (v_i)_{i=0}^k$ and $b = u_0$ and $c = u_h$ are connected by a walk $W_2 = (u_i)_{i=0}^h$, then there is a walk from a to c obtained by concatenating W_1 and W_2 into $W_1W_2 = (w_i)_{i=0}^{k+h}$, where $w_i = v_i$ for $0 \leq i \leq k$ and $w_{k+i} = u_i$ for $1 \leq i \leq h$.

The equivalence class of a vertex v determined by the connectedness relation gives the component of the graph that contains the vertex v .

We can also consider walks in non-simple graphs.

Example 3 Let G be the graph in Figure 4. A walk in G is given by

$$(h, j, j, i, e, f, f) = (b \xrightarrow{h} c \xrightarrow{j} c \xrightarrow{i} b \xrightarrow{e} a \xrightarrow{f} d \xrightarrow{f} a).$$

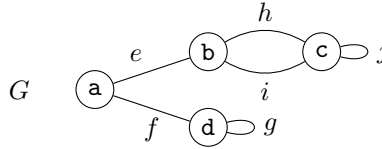


Figure 4: A non simple graph G .

3 Connected graphs

A graph is *connected* if it is non-empty and every two of its vertices are linked by a path in it. A maximal connected subgraph is a *component* of the graph. A graph with more than one component (i.e., that is not connected) is called *disconnected*.

Example 4 The graph G in Figure 5 has three components:

$$H_1 = G[\{a, b, c, d, e\}], \quad H_2 = G[\{f, g, h, i\}] \quad \text{and} \quad H_3 = G[\{j\}].$$

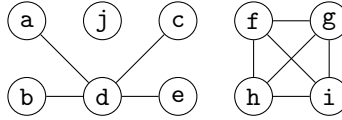


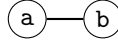
Figure 5: A graph with three components.

Let S_n be the number of different connected graphs on n vertices.

- If $V = \{a\}$, there is only one possible graph, so $S_1 = 1$.



- If $V = \{a, b\}$, the two vertices are connected by an edge, so $S_2 = 1$.



- If $V = \{a, b, c\}$, there are four possible connected graphs, so $S_3 = 4$.



Theorem 1 For every $n \geq 1$

$$n \cdot 2^{\binom{n}{2}} = \sum_{k=1}^n \binom{n}{k} S_k \cdot k \cdot 2^{\binom{n-k}{2}}.$$

Proof. Let us count, in two different ways, all "rooted" graphs on n vertices, i.e., graphs with one particular vertex emphasised.

We know that the number of labelled graphs (connected or not) is $2^{\binom{n}{2}}$. So, the total number of rooted graphs is $n \cdot 2^{\binom{n}{2}}$.

On the other hand, each "root" will appear in a connected component of size k , with $1 \leq k \leq n$. For a fixed k , we have $\binom{n}{k}$ possibilities to select the k vertices, S_k ways of having the component connected, and k ways of selecting the "root" in the connected component; we do not know whether the remaining $n - k$ vertices are connected or not, so we have $2^{\binom{n-k}{2}}$ possibilities for them. ■

Theorem 1 is quite unsatisfactory since it is a recursive formula. To compute, e.g., S_{20} one needs to successively compute $S_1, S_2, S_3, \dots, S_{19}$.

Example 5 We can compute S_4 using Theorem 1, since

$$4 \cdot 2^{\binom{4}{2}} = \binom{4}{1} S_1 \cdot 1 \cdot 2^{\binom{3}{2}} + \binom{4}{2} S_2 \cdot 2 \cdot 2^{\binom{2}{2}} + \binom{4}{3} S_3 \cdot 3 \cdot 2^{\binom{1}{2}} + \binom{4}{4} S_4 \cdot 4 \cdot 2^{\binom{0}{2}}$$

we have, using the known values of S_1, S_2 and S_3 ,

$$4 \cdot 64 = 4 \cdot 1 \cdot 1 \cdot 8 + 6 \cdot 1 \cdot 2 \cdot 2 + 4 \cdot 4 \cdot 3 \cdot 1 + 1 \cdot S_4 \cdot 4 \cdot 1.$$

Hence, $S_4 = 64 - 8 - 6 - 12 = 38$.

Exercise $[S_4]$. Find all connected simple graphs of order 4.

4 Trees

A graph without cycles is called *acyclic*. An acyclic graph is also called a *forest*. A connected forest, i.e., a graph that is both connected and acyclic, is called a *tree*. Vertices of degree 1 in a tree are called *leaves*.

Example 6 The graph G on the left of Figure 6 is not acyclic. The graph H in the centre is a forest. The graph $K_{1,5}$, on the right, is a tree.

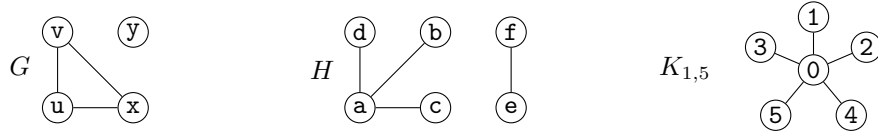


Figure 6: A graph with a cycle (left), a forest (centre), and a tree (right).

The trivial graph has no leaves, since the only vertex has degree 0. Every tree having at least 2 vertices has at least 2 leaves. Indeed, it is enough to consider the endpoints of a longest path.

Theorem 2 Let $G = (V, E)$ be a finite graph s.t. $d_G(v) \geq 2$ for every $v \in V$. Then G contains a cycle.

Proof. The statement is clear if G has a loop or two parallel edges. Thus, WLOG, let us suppose that G is simple.

Let $P = (v_0, v_1, \dots, v_k)$ be a longest path in G . Since $d_G(v_k) \geq 2$, there exists $v \in N_G(v_k)$ with $v \neq v_{k-1}$. If $v \notin P$, then the path $P + \{v\} = (v_0, v_1, \dots, v_k, v)$ would be longer than P , which is a contradiction. Thus, $v = v_i$, for some $0 \leq i \leq k-2$, which implies that $(v_i, \dots, v_{k-1}, v_k)$ is a cycle in G . ■

A graph $G = (V, E)$ is *minimally connected* if it is connected but $G - e$ is disconnected for every $e \in E$. It is *maximally acyclic* if it is acyclic but for any two vertices $u, v \in V$ s.t. $\{u, v\} \notin E$ we have that $G + \{u, v\}$ contains a cycle.

Theorem 3 The following are equivalent for a graph $G = (V, E)$.

1. G is a tree.
2. Any two vertices in V are linked by a unique path in G .
3. G is minimally connected.
4. G is maximally acyclic.

Proof. Exercise.

(Hint: Prove that $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 1.$) ■

Corollary 1 (Euler’s formula for trees) *A connected graph of order n is a tree if and only if it has size $n - 1$.*

The hypothesis of connectedness is necessary.

Example 7 Consider the graphs G, H and $K_{1,5}$ in Example 6. The graph H , a forest but not a tree, has order 6 and size 4. The tree $K_{1,5}$ has order 6 and size 5. The graph G , which is not a forest, has order 4 and size 3.

Theorem 4 (Cayley’s formula) *Let $\#V = n \geq 1$. The number of trees with set of vertices V is n^{n-2} .*

Example 8 The only tree with one vertex is the trivial one. The only tree with two vertices a, b is the one with one edge $\{a, b\}$. If the set of vertices is $\{a, b, c\}$, we have three possible trees (see Figure 7). According to Theorem 4 there are 16 labelled trees on $V = \{a, b, c, d\}$.

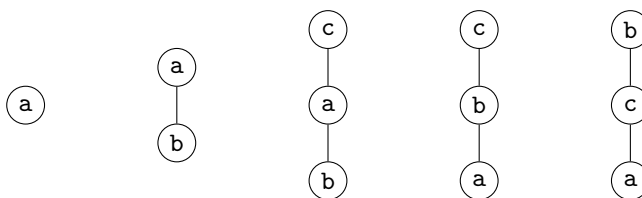


Figure 7: Possible trees on 1, 2 and 3 vertices.

Exercise [Cayley’s formula]. Draw the 16 labelled trees on the set of vertices $\{a, b, c, d\}$. How many of them are isomorphic to $K_{1,3}$? How many to P_3 ?

A *rooted tree* $T(r)$ is a tree $T = (V, E)$ with a specified vertex $r \in V$ called the *root* of T . Given a vertex v in a rooted tree $T(r)$ different from r , we call the vertex in $N_T(v)$ on the unique path between r and v a *parent* of v ; all vertices in this path are called *ancestors* of v . Similarly, a vertex is a *child* of its parent and u is a *descendant* of v if v is an ancestor of u .

The *level* $\ell_T(v)$ of a vertex v in a rooted tree $T(r)$ is defined as the distance $d_T(r, v)$ (i.e., the length of the path from r to v in the tree). The root is the only vertex with level 0. The *height* of a rooted tree is the maximal level.

When dealing with rooted trees, we call *leaf* a vertex that has no children. Hence, in a non-trivial rooted tree the root is not a leaf (in the trivial rooted tree, we have only one vertex that is at the same time the root and a leaf).

Example 9 The graph in Figure 8 is a rooted tree with root r . The vertices a, b, c are children of r . The vertex h is a child of e and a descendant of a (hence e is a parent of h and a an ancestor of h). The vertices d, e, f, g have level 2, while the height of the tree is 3 (h is a vertex of maximal level).

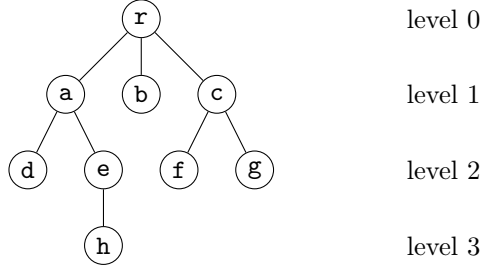


Figure 8: A rooted tree.

An *ordered tree* $T(r, \preceq)$ is a rooted tree $T(r) = (V, E)$ with root r and a partial order $\preceq$ on V giving the order of the children of each vertex (i.e., vertices with different parents are not ordered). We usually represent it by drawing the children of a node from left to right starting from the smallest.

Example 10 Let T be the rooted tree in Example 9 and represented in Figure 8. Then, $T(r, \preceq)$, with the partial order $\preceq$ defined by

$$a \prec b \prec c, \quad d \prec e \quad \text{and} \quad f \prec g$$

is an ordered tree.

5 Adjacency matrix

Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* A_G is the $n \times n$ matrix defined by

$$[A_G]_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}.$$

Clearly A_G is a non-negative symmetric matrix.

Example 11 Let $G = (\{a, b, c, d\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}\})$. The adjacency matrix of G is

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the definition of A_G can be generalised to non-simple graphs by replacing 1 with the number of parallel edges and counting each loop once.

Theorem 5 Let $k \in \mathbb{N}_0$ and let A_G be the adjacency matrix of a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$. Then $[A_G^k]_{i,j}$ is the number of walks of length k with endpoints v_i and v_j in G .

Proof. The case $k = 0$ is trivial since we have $A_G^0 = I_n$, and the only walks of length 0 are the ones connecting a vertex to itself.

Let us prove it by induction on $k \geq 1$.

($k = 1$) It follows directly from the definition of adjacency matrix, since a walk of length 1 is exactly an edge.

($k - 1 \rightarrow k$) The (i, j) -element of A_G^k is given by

$$[A_G^k]_{i,j} = [A_G^{k-1} \cdot A_G]_{i,j} = \sum_{\ell=1}^n [A_G^{k-1}]_{i,\ell} \cdot [A_G]_{\ell,j} = \sum_{\substack{\ell=1 \\ \{v_\ell, v_j\} \in E}}^n [A_G^{k-1}]_{i,\ell}$$

but $[A_G^{k-1}]_{i,\ell}$ is, by induction hypothesis, the number of walks of length $k - 1$ with endpoints v_i and v_ℓ . Since we are summing only the walks that can be prolonged from v_ℓ to v_j (for every possible choice of v_ℓ) we can conclude (see Figure 9).

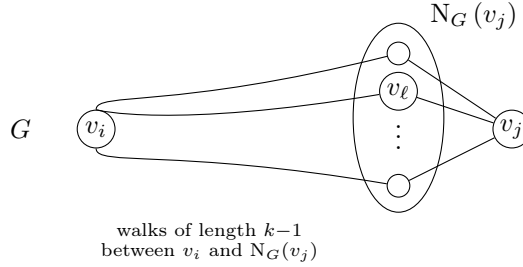


Figure 9: Illustration of the proof of Theorem 5. ■

Recall that a matrix is *positive* if all its entries are strictly positive.

Corollary 2 *A graph $G = (V, E)$ is connected if and only if $\sum_{k=0}^{\#V-1} A_G^k$ is a positive matrix.*

Proof.

($\Rightarrow$) Let $v_i, v_j \in V$. Since G is connected the two vertices are linked in G . Thus, there is a walk of length $k \leq \#V - 1$ in G with endpoints v_i and v_j . Hence, $[A_G^k]_{i,j} \geq 1$.

($\Leftrightarrow$) Let i, j with $1 \leq i, j \leq n$. Since $\left[\sum_{k=0}^{n-1} A_G^k \right]_{i,j} > 0$, there exists ℓ , with $0 \leq \ell \leq n-1$ such that $[A_G^\ell]_{i,j} \geq 1$. Thus, there is a walk of length ℓ in G with endpoints v_i and v_j . ■

Example 12 Let A_G be the adjacency matrix seen in Example 11. We have

$$A_G^2 = \begin{pmatrix} \textcolor{red}{3} & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_G^3 = \begin{pmatrix} 2 & 4 & \textcolor{blue}{4} & 3 \\ 4 & 3 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Indeed, we have, e.g., three walks of length 2 from **a** to **a**, namely

$$(\mathbf{a}, \mathbf{b}, \mathbf{a}), (\mathbf{a}, \mathbf{c}, \mathbf{a}) \quad \text{and} \quad (\mathbf{a}, \mathbf{d}, \mathbf{a});$$

and four walks of length 3 from **a** to **c**, namely:

$$(\mathbf{a}, \mathbf{b}, \mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{c}, \mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{c}) \quad \text{and} \quad (\mathbf{a}, \mathbf{d}, \mathbf{a}, \mathbf{c}).$$

Notice also that

$$\sum_{k=0}^3 A_G^k = I_4 + A_G + A_G^2 + A_G^3 = \begin{pmatrix} 6 & 6 & 6 & 4 \\ 6 & 6 & 5 & 3 \\ 6 & 5 & 5 & 2 \\ 4 & 3 & 2 & 2 \end{pmatrix}$$

is a positive matrix.

6 Isomorphism of graphs

Two graphs $G = (V, E)$ and $H = (U, F)$ are *identical* if $V = U$ and $E = F$.

They are *isomorphic*, denoted $G \sim H$ (or $G \cong H$), if there exists a bijection $\varphi : V \rightarrow U$, called an *isomorphism*, such that for every $u, v \in V$

$$\{u, v\} \in E \quad \Leftrightarrow \quad \{\varphi(u), \varphi(v)\} \in F.$$

Clearly, when $\varphi = \text{id}$, the two graphs are identical.

Example 13 The graphs $G = \left(V, \binom{V}{2} \right)$ and $H = \left(U, \binom{U}{2} \right)$ with $V = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and $U = \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}\}$ are isomorphic (see Figure 10), with bijection, e.g.,

$$\varphi : \begin{cases} \mathbf{a} \mapsto \mathbf{x} \\ \mathbf{b} \mapsto \mathbf{y} \\ \mathbf{c} \mapsto \mathbf{z} \\ \mathbf{d} \mapsto \mathbf{t} \end{cases}$$

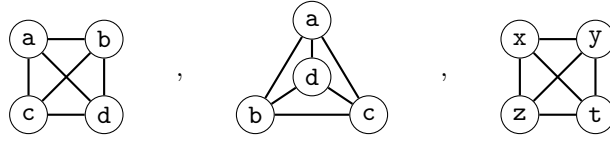


Figure 10: Three isomorphic graphs (the first two are identical).

Thus, two graphs are isomorphic if they are "the same" up to relabelling the vertices. Clearly, degree and distance are preserved under isomorphism.

Proposition 1 *Let $G = (V, E)$ and $H = (U, F)$ be two isomorphic graphs with isomorphism φ . Then, for every $u, v \in V$ we have*

1. $d_G(v) = d_H(\varphi(v))$,
2. $d_G(u, v) = d_H(\varphi(u), \varphi(v))$.

Proof.

1. From the definition of isomorphism it follows that $u \in N_G(v)$ if and only if $\varphi(u) \in N_H(\varphi(v))$. Thus, the cardinality of the two sets, i.e., the two degrees $d_G(v)$ and $d_H(\varphi(v))$, coincide.

2. Exercise.

■

Example 14 The graphs represented in Figure 11 are isomorphic. A possible bijection is

$$\varphi : \begin{cases} a \mapsto 1 \\ b \mapsto 4 \\ c \mapsto 2 \\ d \mapsto 5 \\ e \mapsto 3 \\ f \mapsto 6 \end{cases} .$$

One can check, e.g., that $d_G(a) = d_H(1) = 3$, and that $d_G(a, e) = d_H(1, 3) = 2$.



Figure 11: Two isomorphic graphs.

We normally do not distinguish between isomorphic graphs. When dealing with a graph up to isomorphism, we can forget the labels in the drawing.

If two graphs are isomorphic, then clearly they have the same order and the same size. The opposite is not true.

Exercise [Non-isomorphic graphs]. Show that the two unlabelled graphs in Figure 12, which have both order 4 and size 3, are not isomorphic.



Figure 12: Two non-isomorphic graphs with the same order and size.

The graph isomorphism problem, i.e., deciding whether two graphs are isomorphic, belongs to $\mathcal{NP}$. It is not known whether it belongs to $\mathcal{P}$ or $\mathcal{NP}$ -complete.