Elementary Introduction to Graph Theory

(01YEIG 2026/2027)

Lecture 7

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Solutions to exercises from the previous lecture

Solution of Exercise [Minimum spanning tree]. Let us order the edges of G according to their weights: $e_1 < e_2 < \cdots < e_{10}$ (this order is not unique). The label and weight of each edge are shown on the left of Figure 1. The minimum spanning tree obtained using Kruskal's algorithm is shown on the right of the same figure.

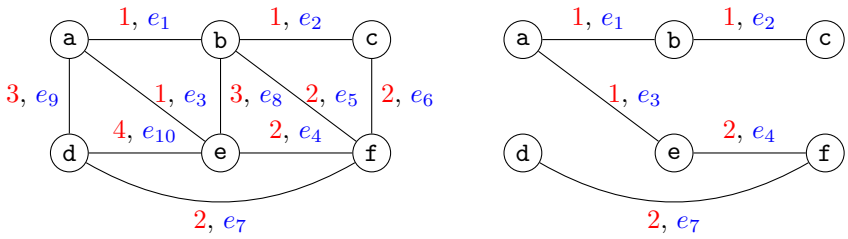


Figure 1: A weighted graph and one of its minimum spanning trees.

Solution of Exercise [Trick-or-Treat Spooky Problem (TSP)]. An optimal route for the graph is given by the cycle $(\mathtt{H}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e})$ of weight 14 (see Figure 2).

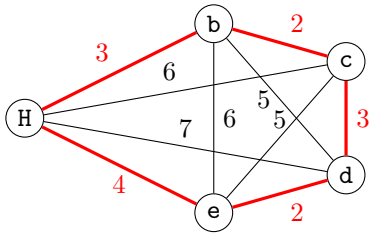


Figure 2: A plan of the neighbourhood.

1 Euler tours (continued)

Example 1 Consider the graph in Figure 3. The graph is Eulerian.

Using the Euler Theorem (see left of the figure) we can construct an Euler tour by starting from a and combine the two closed walks $W_r = (\mathtt{a} \xrightarrow{e_1} \mathtt{b} \xrightarrow{e_2} \mathtt{c} \xrightarrow{e_3} \mathtt{a})$ and $W_g = (\mathtt{c} \xrightarrow{e_4} \mathtt{d} \xrightarrow{e_5} \mathtt{e} \xrightarrow{e_6} \mathtt{c})$, into

$$(e_1, e_2, e_4, e_5, e_6, e_3).$$

Or, we can use the algorithm based on Tarry's traversal (see right of the figure) and, from the closed walk $(\mathtt{a} \xrightarrow{e_1} \mathtt{b} \xrightarrow{e_2} \mathtt{c} \xrightarrow{e_3} \mathtt{a} \xrightarrow{e_3} \mathtt{c} \xrightarrow{e_4} \mathtt{d} \xrightarrow{e_5} \mathtt{e} \xrightarrow{e_6} \mathtt{c} \xrightarrow{e_6} \mathtt{e} \xrightarrow{e_5} \mathtt{d} \xrightarrow{e_4} \mathtt{c} \xrightarrow{e_2} \mathtt{b} \xrightarrow{e_1} \mathtt{a})$, obtain the Euler tour

$$(e_3, e_6, e_5, e_4, e_2, e_1).$$

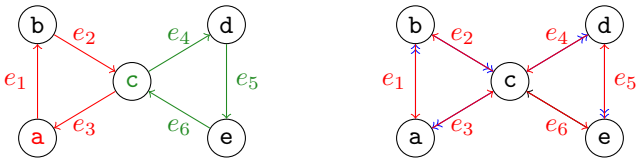


Figure 3: Two constructions to obtain an Euler tour.

Recall that the number of vertices with odd degrees in a graph is even. Therefore, in particular, it cannot be 1.

Theorem 1 *A connected graph has an Euler trail if and only if the number of vertices with odd degree is at most* 2 *(i.e., either* 0 *or* 2*).*

Example 2 The graphs G and K in Figure 4 are not Eulerian. Indeed, they are connected but not even, since, e.g., $d_G(\mathtt{e}) = 3$ and $d_K(A) = 5$.

The graph G has an Euler trail, since only `d` and `e` have odd degrees.

The graph K does not have an Euler trail since it has 4 vertices of odd degree.



Figure 4: Two non-Eulerian graphs.

2 Hamiltonian cycles

A problem analogous to finding an Euler tour in a graph is that of finding a path or a cycle that visits every vertex exactly once.

We call *the Hamiltonian path* and *the Hamiltonian cycle*, respectively, a spanning path and a spanning cycle in a graph G.

Thus, every Hamiltonian path in $G = (V, E)$ has length $\#V - 1$ and every Hamiltonian cycle has length $\#V$.

A graph containing a Hamiltonian path is called *traceable*. A graph containing a Hamiltonian cycle is called *Hamiltonian*. Clearly, every Hamiltonian graph is also traceable.

Example 3 The graph in Figure 5 is Hamiltonian. Indeed, a Hamiltonian path is shown on the left, and a Hamiltonian cycle on the right of the figure.



Figure 5: A Hamiltonian path (on the left) and a Hamiltonian cycle (on the right).

There is a major difference between Eulerian and Hamiltonian problems. While determining whether a given graph has an Euler tour can be done efficiently, determining whether a graph has a Hamiltonian cycle is $\mathcal{NP}$-complete. No simple characterisation is known or even expected to exist for arbitrary graphs.

Path exchanges. A natural way to look for a Hamiltonian path is by extending a path as much as possible. Suppose that we found a path $P = (v_1, \ldots, v_k)$ in a graph G and let $v \in \mathrm{N}_G(v_k) \setminus \{v_k\}$ (we want to avoid loops).

- If v is not in P, i.e., if $v \neq v_i$ for all $1 \leq i \leq k$, then we can extend P to the longer path
$$P' = (v_1, \ldots, v_k, v).$$
- If v is in the path but $v \neq v_{k-1}$, i.e., $v = v_i$ for some $1 \leq i \leq k-2$, then we can obtain a path of the same length by replacing the edge $\{v_i, v_{i+1}\}$ with the edge $\{v_k, v_i\}$, i.e., considering the path (see Figure 6)
$$P'' = (v_1, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}).$$

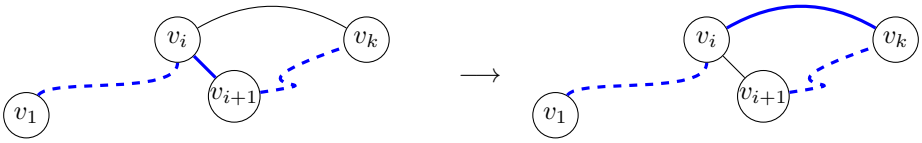


Figure 6: A path exchange.

Cycle exchange. In a similar way, let us suppose that we have a cycle $C = (v_1, \ldots, v_k)$ having two vertices v_i, v_j not consecutive in C, i.e., with $|i-j| > 1$ and such that both $\{v_i, v_j\}$ and $\{v_{i+1}, v_{j+1}\}$ are edges of the graph (we consider $i+1, j+1 \pmod{k}$). Then we can obtain a new cycle
$$C' = (C \setminus \{\{v_i, v_{i+1}\}, \{v_j, v_{j+1}\}\}) + \{\{v_i, v_j\}, \{v_{i+1}, v_{j+1}\}\}$$
having the same length as C (see Figure 7).

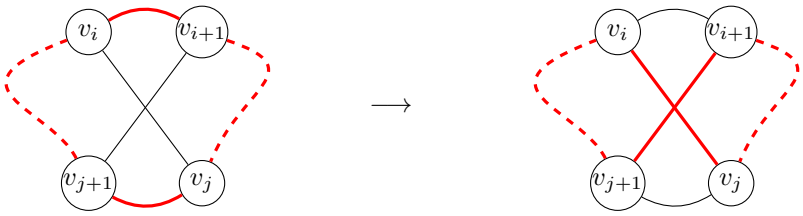


Figure 7: A cycle exchange.

There are several known sufficient conditions for the existence of a Hamiltonian cycle in a graph.

Every complete graph of order at least 3 is Hamiltonian, since a Hamiltonian cycle can be obtained by selecting all vertices one by one in an arbitrary order.

Example 4 The complete graph K_5 over set of vertices $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ has, e.g., the Hamiltonian cycle: $(\mathtt{b}, \mathtt{c}, \mathtt{a}, \mathtt{e}, \mathtt{d})$ (see Figure 8).

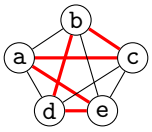


Figure 8: A Hamiltonian cycle in K_5.

When we have fewer edges, finding a Hamiltonian cycle is harder. Dirac proved what is the minimum degree that a graph must have to guarantee the existence of a Hamiltonian cycle.

Theorem 2 (Dirac (1952)) *Every simple graph G of order $n \geq 3$ such that $\delta(G) \geq \frac{n}{2}$ is Hamiltonian.*

Dirac's Theorem has the best possible bound on $\delta(G)$ to ensure the existence of a Hamiltonian cycle. We cannot, e.g., replace $\frac{n}{2}$ with $\lfloor \frac{n}{2} \rfloor$.

Example 5 Let n be an odd number and G be the union of two complete graphs $K_{\lceil \frac{n}{2} \rceil}$ sharing exactly one vertex w (see Figure 9, where $n = 7$). Then, $\delta(G) = \lfloor \frac{n}{2} \rfloor$, but G cannot have a Hamiltonian cycle as it would have to go through w twice (such a w is also called a *cut vertex*).

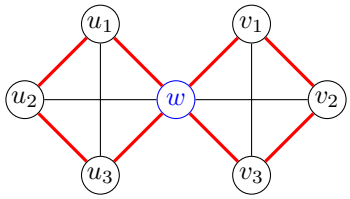


Figure 9: A closed walk visiting all vertices passing through w twice.

We can find other sufficient conditions.

Theorem 3 (Ore (1960)) *Let $G = (V, E)$ be a simple graph and let $u, v \in V$ not adjacent and such that $d_G(u) + d_G(v) \geq \#V$. Then, G is Hamiltonian if and only if $G + \{u, v\}$ is Hamiltonian.*

Proof.

($\Rightarrow$) If G has a Hamiltonian cycle, then clearly the same cycle is also in $G + \{u, v\}$.

($\Leftarrow$) If $G + \{u, v\}$ has a Hamiltonian cycle C, then we can find a Hamiltonian cycle C' by using a cycle exchange (Exercise).

■

The *closure* of a (simple) graph $G = (V, E)$ is the graph obtained from G by recursively adding edges $\{u, v\}$ if $\{u, v\}$ is not an edge in the graph and the sum of the two degrees is at least $\#V$, until all vertices of such a form are adjacent. One can prove that the order in which the edges are added does not change the final result (why?).

Example 6 Let G be the graph of Example 3. The closure of G is the complete graph K_5 (see Figure 10).

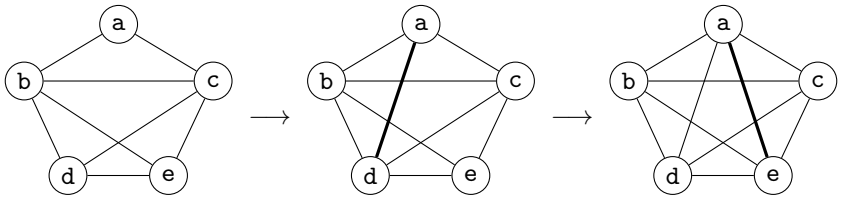


Figure 10: Closure of a graph.

A consequence of Theorem 3 is the following.

Corollary 1 *A simple graph is Hamiltonian if and only if its closure is Hamiltonian.*

There exist other sufficient conditions for a graph to have a Hamiltonian cycle. Most of them require the graph to have "enough" edges.

Theorem 4 (Pósa (1962)) *Let $G = (V, E)$ be a simple graph of order n realised by a graphic sequence $(d_i)_{i=1}^n$ with $d_1 \leq d_2 \leq \cdots \leq d_n$. If for every $k < \frac{n}{2}$ we have $d_k > k$, then G is Hamiltonian.*

Example 7 The Hamiltonian graph G of Example 3 is realised by the graphic sequence $(d_i)_{i=1}^5 = (2, 3, 3, 4, 4)$. For $k < \frac{5}{2}$, i.e., for $k = 1, 2$, we have $d_1 = 2 > 1$ and $d_2 = 3 > 2$.

Theorem 5 (Chvátal (1972)) *Let $G = (V, E)$ be a simple graph of order $n \geq 3$ realised by the graphic sequence $(d_i)_{i=1}^n$ with $d_1 \leq d_2 \leq \cdots \leq d_n$. If for every $k < \frac{n}{2}$ either $d_k > k$ or $d_{n-k} \geq n - k$, then G is Hamiltonian.*

Note that the condition of the previous theorem is equivalent to ask that for every $k < \frac{n}{2}$

$$d_k \leq k \;\Rightarrow\; d_{n-k} \geq n - k.$$

Example 8 Consider the graph shown in Figure 11. This graph has order 7 and it is realised by the graphic sequence $(d_i)_{i=1}^7 = (2, 3, 3, 5, 5, 5, 5)$. One has $d_3 = 3 \leq 3$ but $d_4 = 5 \geq 4$. Moreover, $d_1 = 2 > 1$ and $d_2 = 3 > 2$. So, the graph

satisfies Chvátal's condition but not Pósa's condition. A Hamiltonian cycle is shown in red in the same figure.

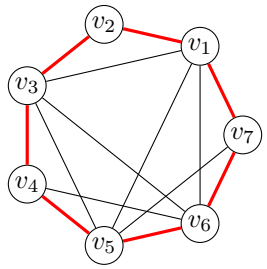


Figure 11: A Hamiltonian cycle in a graph.

Note that the conditions shown above are sufficient but not necessary. A graph may be Hamiltonian even without satisfying any of them.

Example 9 Consider the cycle C_5 (see Figure 12). Its degree sequence is $(d_i)_{i=1}^5 = (2, 2, 2, 2, 2)$. The graph is Hamiltonian, since the graph itself is a Hamiltonian cycle. However, $\delta(C_5) = 2 < \frac{5}{2}$ (Dirac's condition), and for $k = 2 < \frac{5}{2}$ we have $d_2 = 2 \leq 2$ (Pósa's condition) and $d_{5-2} = d_3 = 2 < 3$ (Chvátal's condition).

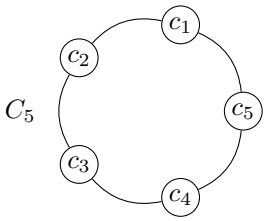


Figure 12: The graph C_5.

Exercise [Petersen graph]. Let us consider the two graphs in Figure 13 (the graph on the right is called *Petersen graph*).

1. Do the graphs have Eulerian trails?
2. Do the graphs have Eulerian tours?
3. Do the graphs have Hamiltonian paths?
4. Do they have Hamiltonian cycles?
5. Do they satisfy Dirac/Pósa/Chvátal conditions?

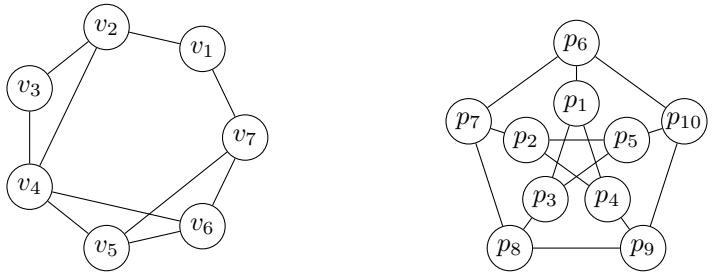


Figure 13: Two graphs (on the right the Petersen graph).

3 Networks

Consider the task of modelling a transportation network with one source s – i.e., one point from where all merchandise leaves – and one sink t – i.e., one point to where all merchandise arrives – in which the amount of flow through a given link between two nodes is subject to a certain capacity of that link. Our aim is to determine the maximum flow through the network from s to t.

Formally, let $D = (V, A)$ be a directed graph.

A *network* with *topology* D is defined as $N = (D, s, t, c)$, where $s, t \in V$ are two distinguished vertices called, respectively, the *source* (producer) and the *sink* (consumer) of the network; and $c : A \to \mathbb{R}_0^+$ is a real-valued function called the *capacity function* of the network N. The value $c(a)$ of an arc $a \in A$ will also be called the *capacity* of a.

In our examples, we will always consider, for simplicity, $c : A \to \mathbb{N}_0$. Also, note that in some contexts it may be useful to allow arcs of infinite capacity (we will not do that, though!).

A vertex $v \in V \setminus \{s, t\}$ is called an *intermediate vertex*, and sometimes we denote the set of all such vertices as I_N.

Example 10 Let $N = (D, \mathtt{s}, \mathtt{t}, c)$ be the network shown in Figure 14. The vertex $\mathtt{s}$ is the source of N, while $\mathtt{t}$ is its sink. The set of intermediate vertices is $I_N = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}\}$. Each arc a is labeled with its capacity $c(a)$.

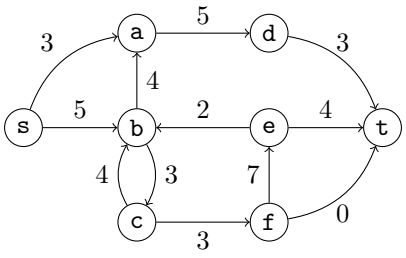


Figure 14: A network.

Let $N = (D, s, t, c)$ be a network, and let $S \subseteq V$ be such that $s \in S$ and $t \notin S$. Let $\bar{S} = V \setminus S$. The pair $(S, \bar{S})$ is called a *cut* in N.

The *capacity* of a cut $(S, \bar{S})$ is defined as

$$c(S, \bar{S}) = \sum_{\substack{a^- \in S \\ a^+ \in \bar{S}}} c(a).$$

Note that we consider only edges "leaving" S.

A cut in a network N is called a *minimum cut* if no cut in N has a smaller capacity.

Example 11 Let N be the network in Example 10 and $S = \{\mathtt{s}, \mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Hence, $\bar{S} = \{\mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{t}\}$. The cut $(S, \bar{S})$ in the network is shown in Figure 15. Its capacity is $8 = 5+3$. Indeed, we consider the sum of the capacities of arcs $(\mathtt{a}, \mathtt{d})$ and $(\mathtt{c}, \mathtt{f})$. Note that we are not considering the arc $(\mathtt{e}, \mathtt{b})$ since it is not leaving from S and arriving to $\bar{S}$.

Similarly, let $T = \{\mathtt{s}, \mathtt{a}, \mathtt{b}, \mathtt{d}\}$. The cut $(T, \bar{T})$ has capacity 6, and one can check that it is a minimum cut.

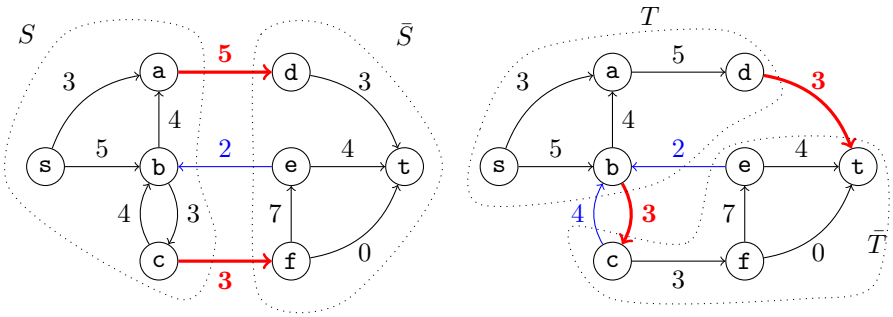


Figure 15: A network with two cuts.

4 Flows

Let $N = (D, s, t, c)$ be a network with topology $D = (V, A)$.

A function $f : A \to \mathbb{R}_0^+$ is called a *flow* in N if it satisfies the two following conditions:

(F1) for every $a \in A$, $0 \leq f(a) \leq c(a)$, (*capacity constraint*)

(F2) for every $v \in I_N$, $\sum_{a^-=v} f(a) = \sum_{a^+=v} f(a)$. (*conservation condition*)

The second condition requires that the *net flow* out of any intermediate vertex be zero. That is, for every intermediate vertex, the total incoming flow equals the total outgoing flow.

Every network has at least one flow. Indeed, it suffices to consider the *zero flow* defined as $f(a) = 0$ for every $a \in A$.

An arc a is called f-positive if $f(a) > 0$. It is f-*unsaturated* if $f(a) < c(a)$ and f-*saturated* if $f(a) = c(a)$.

Example 12 Let N be the network in Example 10. A flow f in N is shown in Figure 16, where we label each arc a with $f(a)/c(a)$. The arc $(\mathtt{b}, \mathtt{c})$ is f-saturated, while the arc $(\mathtt{c}, \mathtt{b})$ is f-unsaturated.

Moreover, one has $\sum_{a^+=\mathtt{b}} f(a) = 3 + 1 + 0 = 1 + 3 = \sum_{a^-=\mathtt{b}} f(a)$.

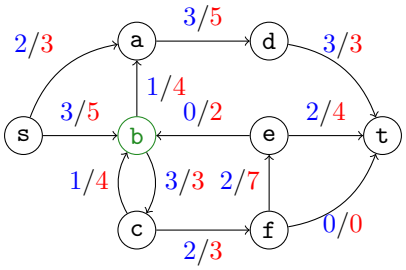


Figure 16: A flow in a network.