Elementary Introduction to Graph Theory

(01YEIG 2026/2027)

Lecture 8



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PDF available at the address:
dolcefra.pages.fit/ens/2627/YEIG-lecture-08.pdf

Solutions to exercises from the previous lecture

Solution of Exercise [Petersen graph].

- The degrees of the vertices of the first graph are:

  $$d(v_1) = d(v_3) = 2, \quad d(v_2) = d(v_5) = d(v_6) = d(v_7) = 3, \quad \text{and} \quad d(v_4) = 4.$$

  There are more than 2 vertices with odd degree, so the graph **cannot have any Euler tour** (**nor Euler trail**).

  The minimum degree is $2 < \frac{7}{2}$, so **Dirac's condition is not satisfied**.

  The degree sequence of the graph is $(d_i)_{i=1}^7 = (2, 2, 2, 3, 3, 3, 4)$. Since $d_2 = 2 \not> 2$, **Pósa's condition is not satisfied**. Since $d_{7-2} = d_5 = 3 \not\geq 5$ **Chvátal's condition is not satisfied** either.

  Nevertheless, the graph has a Hamiltonian cycle (and thus a Hamiltonian path): $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ (see right of Figure 1).

- The second graph, the Petersen graph, is a cubic graph, so it has more than 2 vertices with odd degrees and **cannot have any Euler trail or Euler tour**.

The graphic sequence realising it is $(d_i)_{i=1}^{10} = (3,3,3,3,3,3,3,3,3,3)$ and it is easy to check that the minimum degree is $3 < \frac{10}{2}$, that $d_3 = 3 \not> 3$ and that $d_{10-3} = d_7 = 3 \not> 7$. So **Dirac's condition, Pósa's condition and Chvátal's condition are not satisfied**.

A Hamiltonian path is: $(p_1, p_3, p_5, p_2, p_4, p_9, p_{10}, p_6, p_7, p_8)$. The graph has no Hamiltonian cycle.

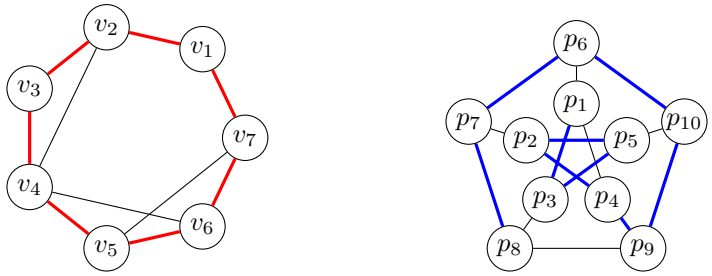


Figure 1: Two graphs (on the right the Petersen graph).

1 Maximum flow

We want to determine the maximum amount of flow through a network. How is this measured?

Let f be a flow in a network $N = (D, s, t, c)$. The *value* of f is

$$\mathrm{val}(f) = \sum_{a^- = s} f(a) - \sum_{a^+ = s} f(a).$$

The value of a flow is basically the net amount of flow that leaves the source s. Note that one also has

$$\mathrm{val}(f) = \sum_{a^+ = t} f(a) - \sum_{a^- = t} f(a)$$

(Prove it!).

A flow f in N is called *maximum* if $\mathrm{val}(f) \geq \mathrm{val}(f')$ for every flow f' in N.

The following theorem states that the value of a flow can be determined not only at the source (or at the sink), but also "on the cut".

Theorem 1 (Net flow across a cut) *Let $N = (D, s, t, c)$ be a network, f be a flow in N and $(S, \bar{S})$ be a cut in N. Then,*

$$val(f) = \sum_{\substack{a^- \in S \\ a^+ \in \bar{S}}} f(a) - \sum_{\substack{a^- \in \bar{S} \\ a^+ \in S}} f(a).$$

Consequently

$$val(f) \leq \sum_{\substack{a^- \in S \\ a^+ \in \bar{S}}} c(a) = c(S, \bar{S}).$$

Hence, the value of a flow in N is never greater than the smallest capacity of a cut in N. In 1956, Ford and Fulkerson, and independently Elias et al., proved that this upper bound is always attained by some flow.

Theorem 2 (Max-Flow Min-Cut) *In every network, the value of a maximum flow equals the minimum capacity of a cut.*

Example 1 Let N be the network on the left of Figure 2. The flow f (shown in blue) in the figure not a maximum flow since its value is $5 = 2 + 3$, while the capacity of a minimum cut in the network is 6.

On the other hand, the flow $\tilde{f}$ shown on the right of the same figure has value 6 and is maximum.

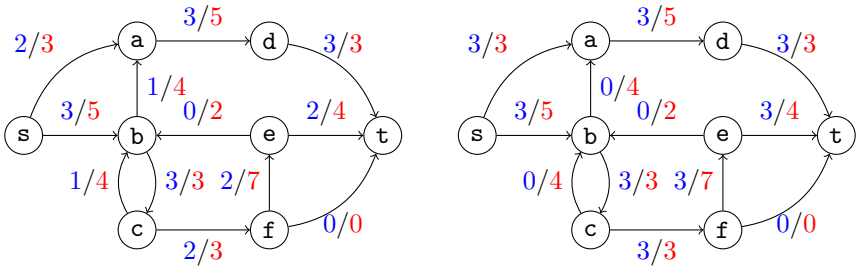


Figure 2: Two flows in a network.

Theorem 2 is nowadays the cornerstone of a standard proof technique in graph theory (in areas not directly connected to flows in networks, e.g., matching or connectivity). It is, incidentally, a corollary of a theorem which characterises maximum flows in a network and which we will state later.

2 Semi-paths

Let $D = (V, A)$ be a directed graph. A sequence $(v_i)_{i=0}^k$ of distinct vertices is called a *semi-path* in D if for each $1 \leq i \leq k$ either $(v_{i-1}, v_i) \in A$ or $(v_i, v_{i-1}) \in A$. In other words, P is a semi-path in D if it is a path in the underlying graph of D. So, a semi-path in a directed graph concerns only the existence of an edge, not its actual direction.

Let $N = (D, s, t, c)$ be a network, f a flow in N, and P a semi-path in D with one endpoint equal to s. Given an arc a in P we define its *residual capacity*

with respect to P, denoted $r(a)$, as

$$r(a) = \begin{cases} c(a) - f(a) & \text{if } a = (v_i, v_{i+1}) \in P \text{ for some } 0 \leq i < k \\ f(a) & \text{if } a = (v_{i+1}, v_i) \in P \text{ for some } 0 \leq i < k \end{cases}.$$

We call an arc of the form (v_i, v_{i+1}) a *forward arc* of P, and an arc of the form (v_{i+1}, v_i) a *backward arc* of P.

The *residual capacity* of the semi-path P is defined as $r(P) = \min_{a \in P} r(a)$.

A semi-path P starting at s is f-*admissible* if $r(P) > 0$. Equivalently, every forward arc in P is f-unsaturated and every backward arc of P is f-positive. An f-admissible semi-path (starting at s and) ending at t is called f-*augmenting*.

Intuitively, an f-augmenting semi-path P is a semi-path that is not being used to its full capacity. On a forward arc, the flow may be increased, while on a backward arc it may be decreased.

Its residual capacity is the largest amount by which the flow f can be augmented along P without violating the capacity constraints.

Example 2 Let N and f be as in Example 1. Consider the two semi-paths $P_1 = (\mathtt{s}, \mathtt{b}, \mathtt{e}, \mathtt{t})$ and $P_2 = (\mathtt{s}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{e}, \mathtt{t})$ in N, with $(\mathtt{a}, \mathtt{b})$ and $(\mathtt{b}, \mathtt{c})$ taken as a backward arc in P_2 (see Figure 3). The residual capacity of P_1 is

$$r(P_1) = \min\{5 - 3,\ 0,\ 4 - 2\} = 0.$$

The residual capacity of P_2 is

$$r(P_2) = \min\{3 - 2,\ 1,\ 1,\ 3 - 2,\ 7 - 2,\ 4 - 2\} = 1 > 0.$$

P_1 is not f-augmenting, while P_2 is f-augmenting.

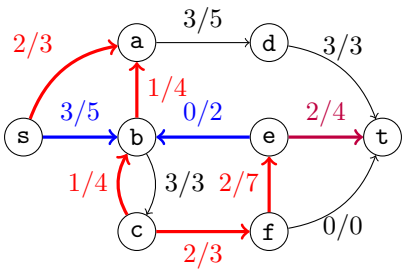


Figure 3: Two semi-paths in a network.

3 Augmenting a flow

If a network N contains an f-augmenting semi-path from s to t, then f is not a maximum flow.

Proposition 1 *Let $N = (D, s, t, c)$ be a network and f a flow in N. If there exists an f-augmenting semi-path P from s to t, then f is not a maximum flow.*

More precisely, the function $\tilde{f}$ defined as

$$\tilde{f}(a) = \begin{cases} f(a) + r(P) & \text{if } a \text{ is a forward arc of } P, \\ f(a) - r(P) & \text{if } a \text{ is a backward arc of } P, \\ f(a) & \text{if } a \text{ is not on } P \end{cases}$$

is a flow in N such that $val(\tilde{f}) = val(f) + r(P)$.

Proof. First, check that $\tilde{f}$ satisfies conditions (F1) and (F2).

(F1) For every forward arc a we have $r(P) \leq c(a) - f(a)$, and hence

$$\tilde{f}(a) = f(a) + r(P) \geq f(a) \geq 0$$

and

$$\tilde{f}(a) = f(a) + r(P) \leq f(a) + c(a) - f(a) = c(a).$$

For every backward arc a we have $r(P) \leq f(a)$, and hence

$$\tilde{f}(a) = f(a) - r(P) \geq 0$$

and

$$\tilde{f}(a) = f(a) - r(P) \leq f(a) \leq c(a).$$

Therefore, for every arc a we have $0 \leq \tilde{f}(a) \leq c(a)$.

(F2) At every intermediate vertex v of P, the two incident arcs of P are modified in such a way that the change in total incoming flow equals the change in total outgoing flow. Hence, flow conservation is preserved. Therefore,

$$\sum_{a^- = v} \tilde{f}(a) = \sum_{a^+ = v} \tilde{f}(a).$$

Finally, since the net flow leaving s increases by $r(P)$, we have

$$\mathrm{val}(\tilde{f}) = \mathrm{val}(f) + r(P).$$

■

We call the flow $\tilde{f}$ of Proposition 1 the *augmented flow* obtained from f along P.

Example 3 Let $f, \tilde{f}$ be the two flows on the left and right of Figure 2. The flow $\tilde{f}$ is obtained by augmenting f along the f-augmenting semi-path $P_2 = (\mathtt{s}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{e}, \mathtt{t})$ (see also Example 2).

4 Characterisation of maximum flows

How to decide whether a flow is already maximum?

We need some systematic way of searching for f-augmenting semi-paths.

Proposition 2 *Let N be a network and f a flow in it. If there is no f-augmenting semi-path in N, then f is a maximum flow in N.*

Proof. Let us consider the set

$$S = \{v \in V \mid \exists\, f\text{-admissible semi-path from } s \text{ to } v\}\,.$$

Obviously $s \in S$ and, by assumption, $t \notin S$. Thus, $(S, \bar{S})$ is a cut in N.

If $a = (x, y)$ with $x \in S$ and $y \in \bar{S}$, then a must be saturated. Otherwise, an f-admissible semi-path from s to x could be extended along a to reach y. Therefore, $f(a) = c(a)$.

If $a = (y, x)$ with $y \in \bar{S}$ and $x \in S$, then $f(a) = 0$. Otherwise, an f-admissible semi-path from s to x can be extended backward along a to reach y.

Since we proved that the value of f can be computed not only at the source but also "on the cut", finally we have

$$\mathrm{val}(f) = \sum_{\substack{a^- \in S \\ a^+ \in \bar{S}}} f(a) - \sum_{\substack{a^- \in \bar{S} \\ a^+ \in S}} f(a) = \sum_{\substack{a^- \in S \\ a^+ \in \bar{S}}} c(a) = c((S, \bar{S})).$$

Therefore, we have a flow f and a cut $(S, \bar{S})$ such that $\mathrm{val}(f) = c((S, \bar{S}))$. It follows from the Max-Flow Min-Cut Theorem that f is a maximum flow in N. ∎

Note that, from the proof of Proposition 2, we are also able to find a minimum cut.

Combining the results seen so far we have the following result.

Theorem 3 (Ford-Fulkerson) *Let $N = (D, s, t, c)$ be a network and f be a flow in N. Then, f is a maximum flow in N if and only if there is no f-augmenting semi-path in N.*

5 Ford-Fulkerson Algorithm

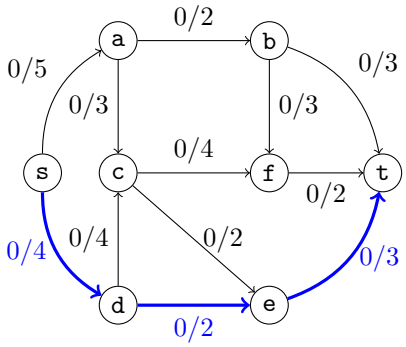
Recall that an f-augmenting semi-path is an f-admissible semi-path from s to t.

The construction of $\tilde{f}$ gives us an algorithm for finding a maximum flow. We can start with a known flow f, e.g., the zero flow, and search for an f-augmenting semi-path. We stop when we find a flow $\tilde{f}$ such that the network does not have any $\tilde{f}$-augmenting semi-paths.

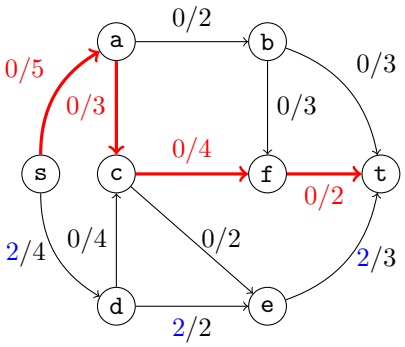
Example 4 Let us consider again the network N in Figure 4a with zero flow f_0.

- The semi-path $P_1 = (\mathtt{s}, \mathtt{d}, \mathtt{e}, \mathtt{t})$ is f_0 -augmenting, since its residual capacity is $r(P_1) = \min\{4-0,\ 2-0,\ 3-0\} = 2 > 0$. The flow f_1 obtained by augmenting f_0 along P_1 is shown in Figure 4b.
- The semi-path $P_2 = (\mathtt{s}, \mathtt{a}, \mathtt{c}, \mathtt{f}, \mathtt{t})$ is f_1-augmenting, since its residual capacity is $r(P_2) = \min\{5-0,\ 3-0,\ 4-0,\ 2-0\} = 2 > 0$. The flow f_2 obtained by augmenting f_1 along P_2 is shown in Figure 4c.
- The semi-path $P_3 = (\mathtt{s}, \mathtt{d}, \mathtt{c}, \mathtt{a}, \mathtt{b}, \mathtt{f}, \mathtt{c}, \mathtt{e}, \mathtt{t})$ is f_2-augmenting, since its residual capacity is $r(P_3) = \min\{4-2,\ 4-0,\ 2,\ 2-0,\ 3-0,\ 2,\ 2-0,\ 3-2\} = 1 > 0$. The flow f_3 obtained by augmenting f_2 along P_3 is shown in Figure 4d.
- The semi-path $P_4 = (\mathtt{s}, \mathtt{a}, \mathtt{c}, \mathtt{f}, \mathtt{b}, \mathtt{t})$ is f_3-augmenting, since its residual capacity is $r(P_4) = \min\{5-2,\ 3-1,\ 4-1,\ 1,\ 3-0\} = 1 > 0$. The flow f_4 obtained by augmenting f_3 along P_4 is shown in Figure 4e.
- The semi-path $P_5 = (\mathtt{s}, \mathtt{a}, \mathtt{b}, \mathtt{t})$ is f_4-augmenting, since its residual capacity is The flow f_5 obtained by augmenting f_4 along P_5 is shown in Figure 4f. $r(P_5) = \min\{5-3,\ 2-1,\ 3-1\} = 1 > 0$.

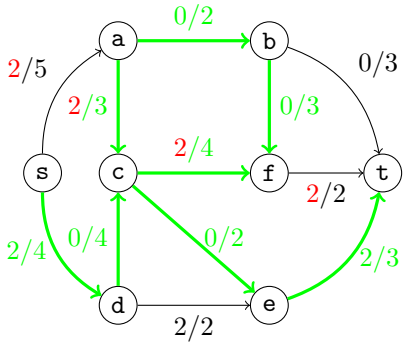
The flow f_5 is a maximum flow. Its value is 7. A minimum cut of the network is given by $(\{\mathtt{s}, \mathtt{a}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}\}, \{\mathtt{b}, \mathtt{t}\})$.



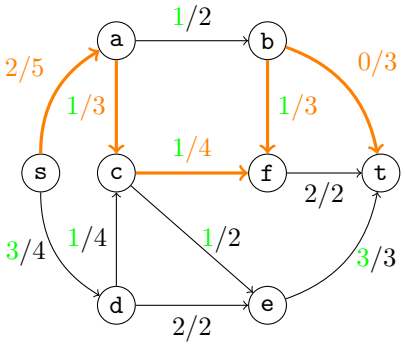
(a) A network with zero flow f_0.



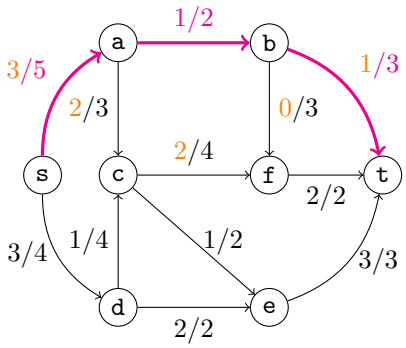
(b) A network with flow f_1.



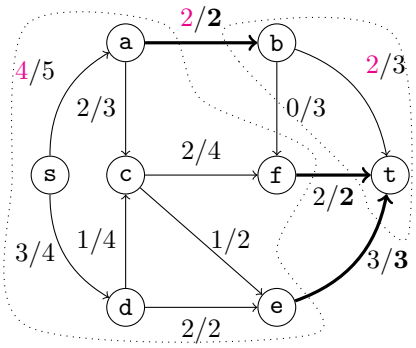
(c) A network with flow f_2.



(d) A network with flow f_3.



(e) A network with flow f_4.



(f) A network with flow f_5.