Interval Exchange Transformations:

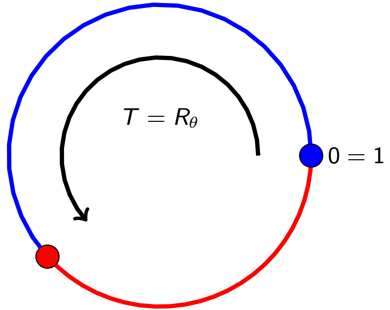
from Symbolic Dynamics to Combinatorics on Words

Francesco Dolce

Theoretical and Computational Algebra 2024

Aveiro, 1 de julho de 2024

Rotations



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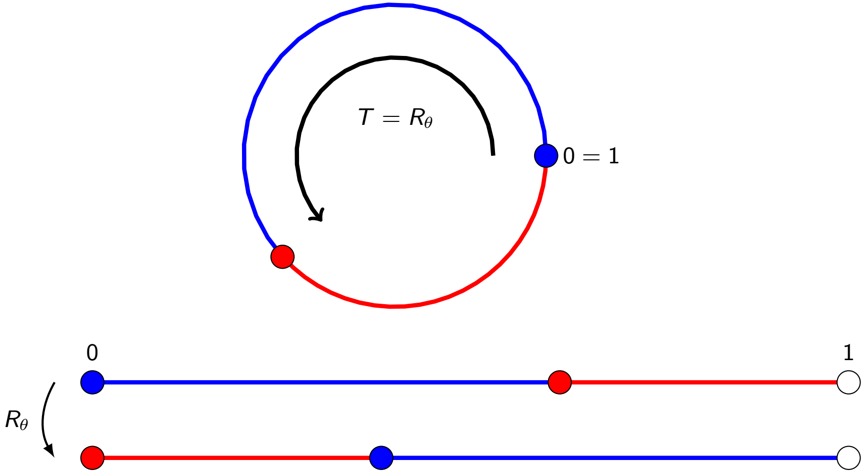

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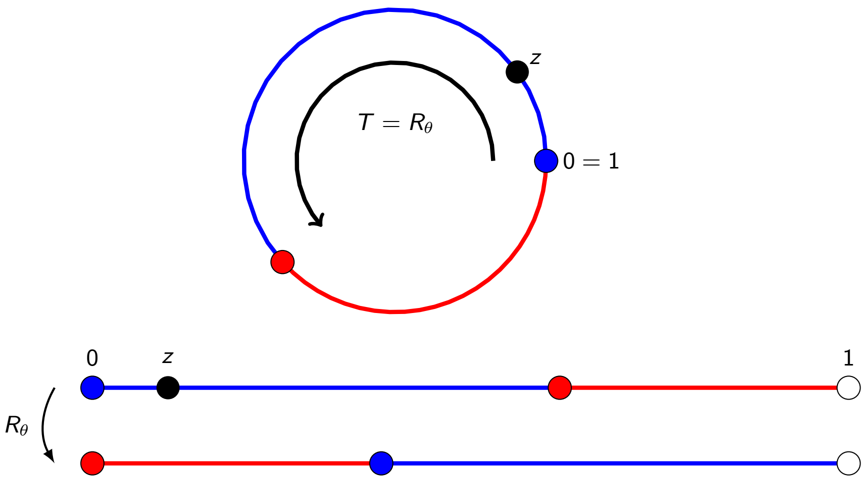

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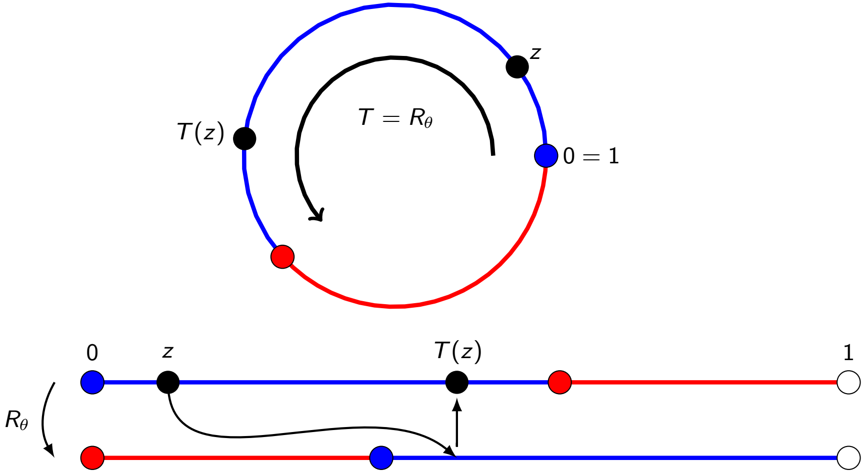

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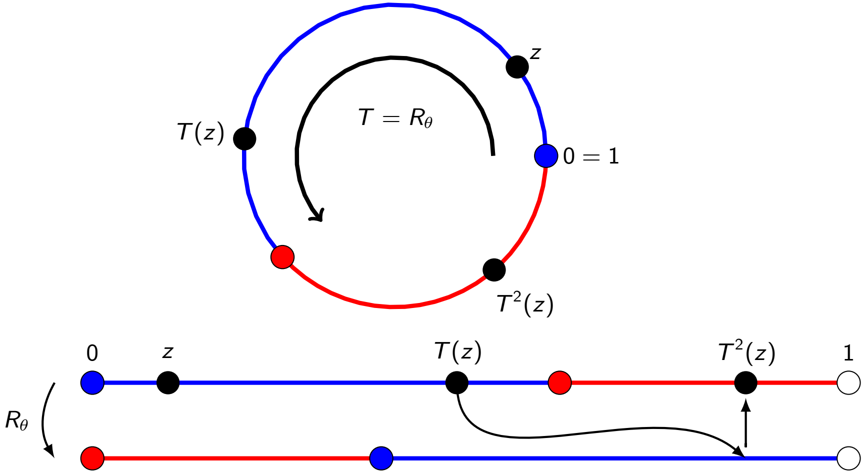

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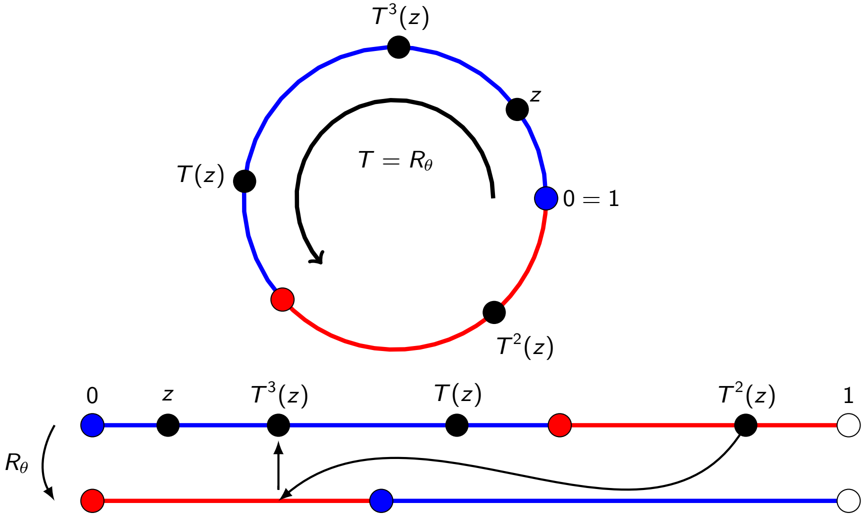
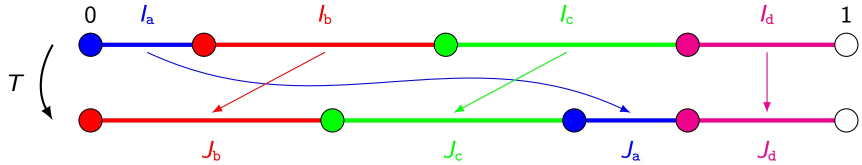


Interval exchanges

Let $(I_a)_{a\in\mathcal{A}}$ and $(J_a)_{a\in\mathcal{A}}$ be two partitions of $[0,1[$ s.t. $|I_a| = |J_a|$ for every $a \in \mathcal{A}$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_a \quad \text{if} \quad z \in I_a.$$



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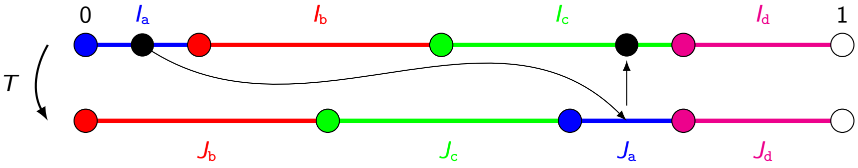

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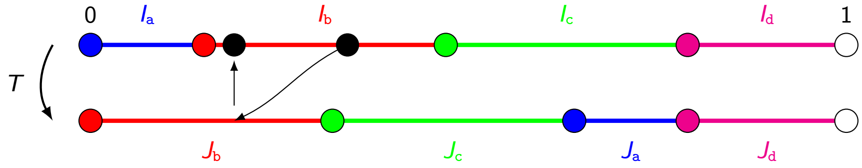

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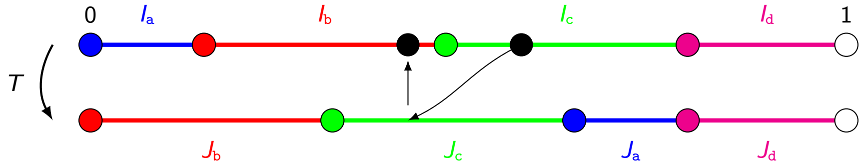

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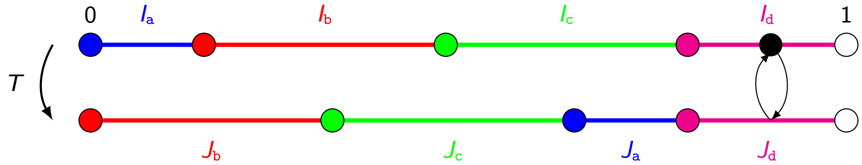

Minimality and regularity

T is *minimal* if for any $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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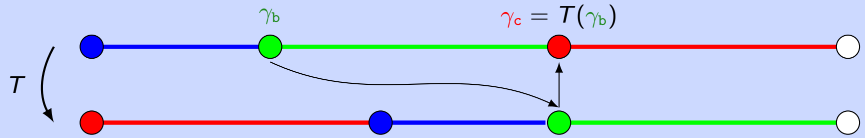
Example (the converse is not true)

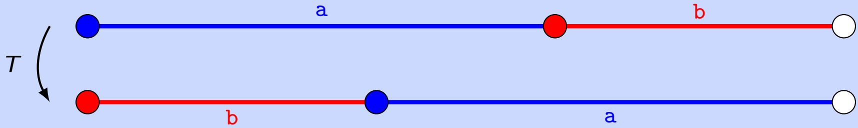


Natural coding

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = a \qquad \text{if } T^n(z) \in I_a$$

Example (Fibonacci)



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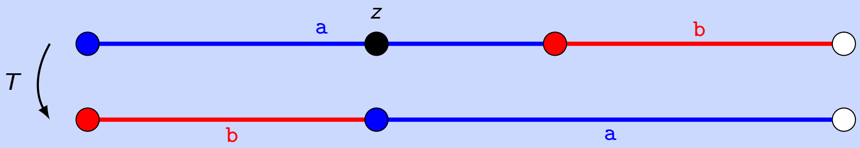

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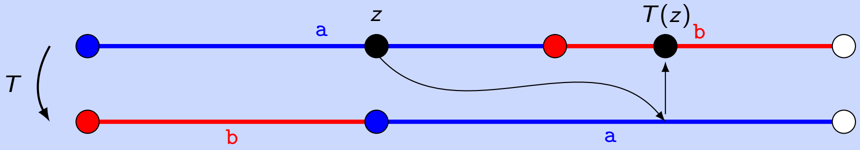

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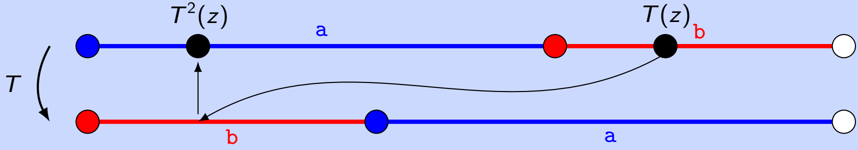

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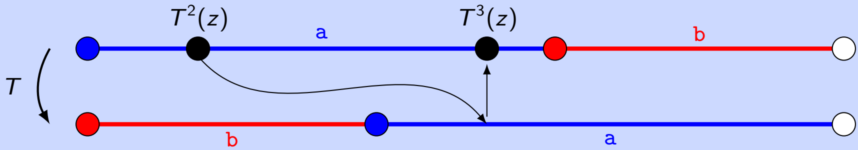

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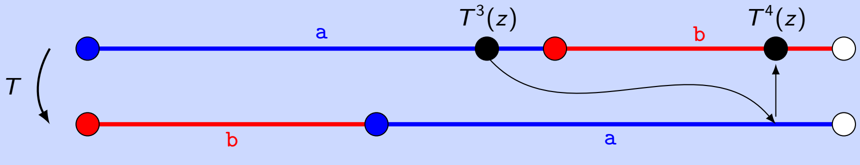

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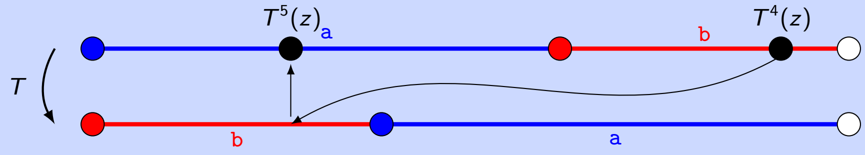

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Interval exchange languages

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathcal{L}(\Sigma_T(z))$ is a (*minimal*, *regular*) *interval exchange language*

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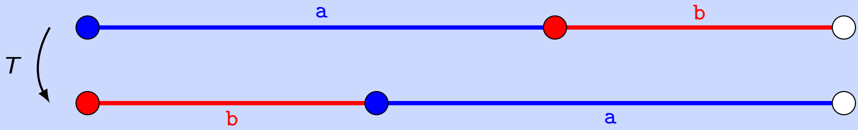

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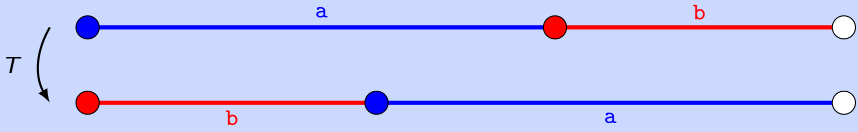
Remark. When T is minimal, $\mathcal{L}(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



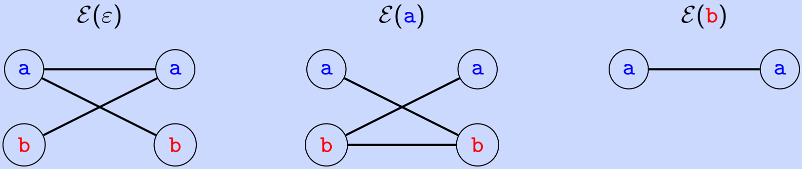
$\mathcal{L}(T) = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \mathtt{aaba}, \ldots\}$

Dendric languages

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Regular interval exchange languages are dendric.

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{aba}, \mathrm{baa}, \mathrm{bab}, \dots\}$)

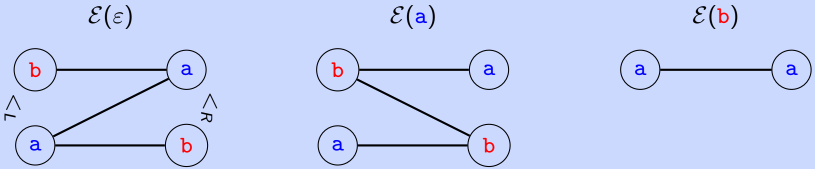


Planar dendric languages

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a language $\mathcal{L}$ and a word $w \in \mathcal{L}$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any (a, b), (c, d) bi-extensions of w, one has

$$a <_L c \qquad \Longrightarrow \qquad b \leq_R d.$$

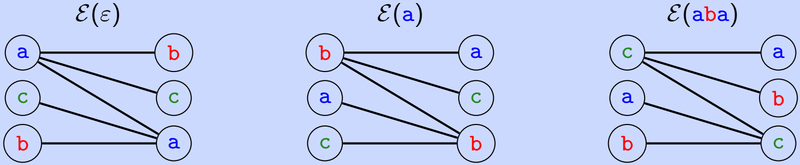
Example (Fibonacci, $\mathtt{b} <_L \mathtt{a}$ and $\mathtt{a} <_R \mathtt{b}$)



Planar dendric languages

Example (Tribonacci, τ : a $\mapsto$ ab, b $\mapsto$ ac, c $\mapsto$ a)

The *Tribonacci* language is **not** planar dendric.



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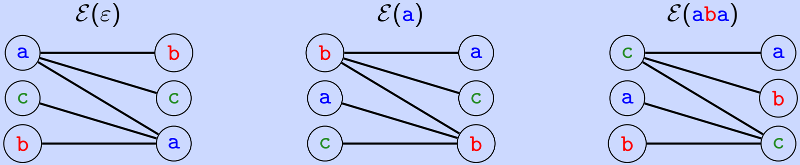
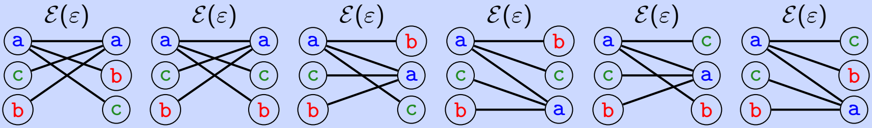


- <u>a $<_L$ c $<_L$ b</u>



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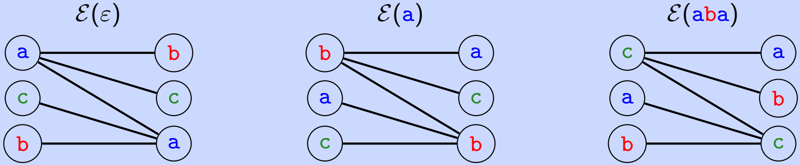
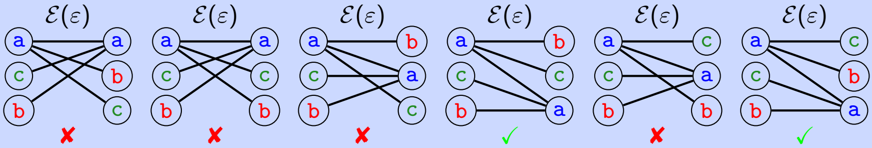


- $a <_L c <_L b \quad \Longrightarrow \quad b <_R c <_R a$ or $c <_R b <_R a$



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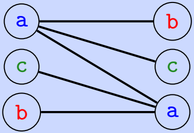
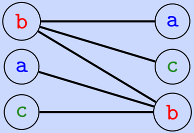
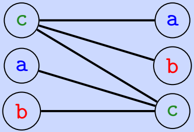


$\mathcal{E}$(a)



$\mathcal{E}$(aba)



- a $<_L$ c $<_L$ b $\quad \Longrightarrow \quad$ b $<_R$ c $<_R$ a or c $<_R$ b $<_R$ a

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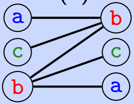

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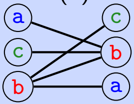

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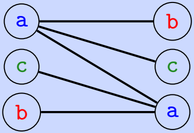

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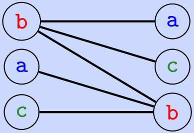

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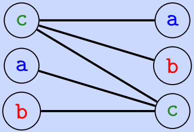

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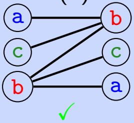

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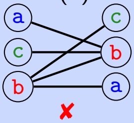

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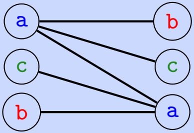

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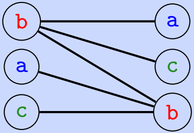

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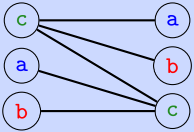

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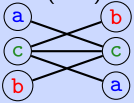

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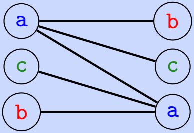

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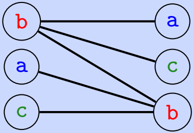

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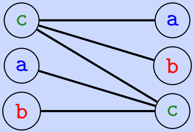

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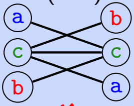
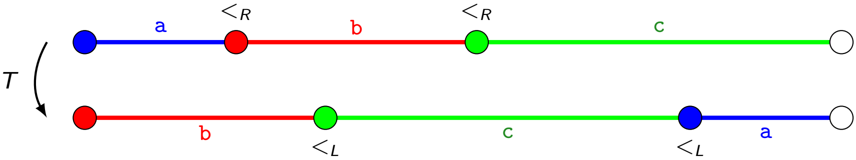


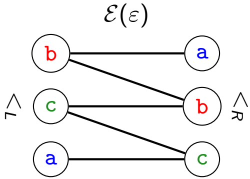
✘

Planar dendric languages

Theorem [Ferenczi, Zamboni (2008)]

A language $\mathcal{L}$ is a regular interval exchange language if and only if it is a recurrent planar dendric language.

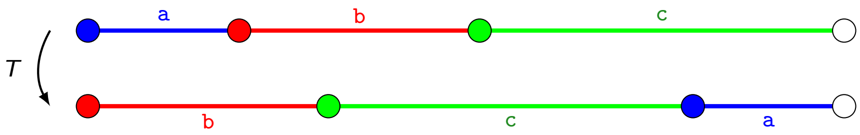




From letters to words

Given a IET T and a word $w = a_0 a_1 \cdots a_m \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \cdots \cap T^{-m}(I_{a_m}) \quad \text{and} \quad J_w = T(I_w)$$



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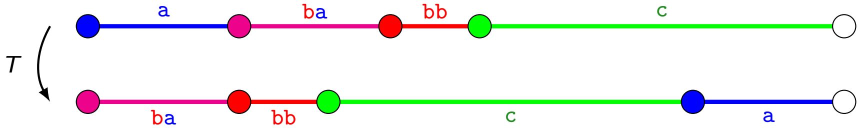
Given a IET T and a word $w = a_0 a_1 \cdots a_m \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \cdots \cap T^{-m}(I_{a_m}) \quad \text{and} \quad J_w = T(I_w)$$



$$I_{ba} = I_b \cap T^{-1}(I_a) \qquad J_{ba} = T(I_b) \cap I_a$$

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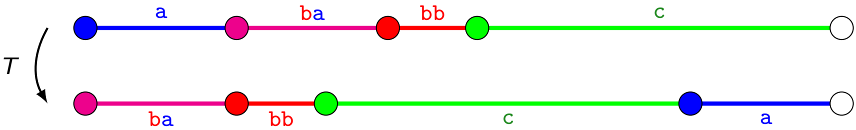

Proposition

If T is minimal, then $w \in \mathcal{L}(T) \Leftrightarrow |J_w| \neq 0$.

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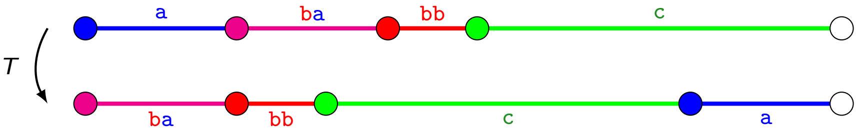


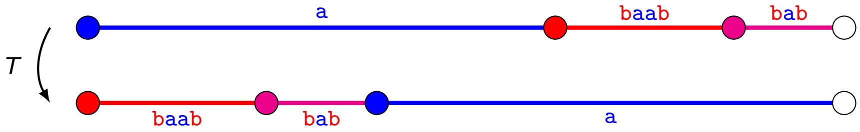
Proposition

- $I_u < I_v$ if and only if $u <_R v$ and u is not a prefix of v.
- $J_u < J_v$ if and only if $u <_L v$ and u is not a suffix of v.

Bifix decoding

Proposition

Let T a minimal IET and X a $\mathcal{L}(T)$-maximal bifix code.
The families $(I_w)_{w \in X}$ and $(J_w)_{w \in X}$ are ordered partitions of $[0,1[$ w.r.t. $<_R$ and $<_L$.



$X = \{\texttt{a}, \texttt{baab}, \texttt{bab}\}$, $\texttt{a} <_R \texttt{baab} <_R \texttt{bab}$ and $\texttt{bab} <_L \texttt{baab} <_L \texttt{a}$

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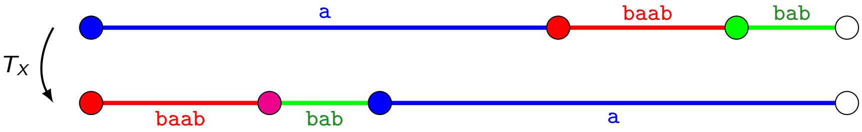

$X = \{$a, baab, bab$\}$, a $<_R$ baab $<_R$ bab and bab $<_L$ baab $<_L$ a

The map $T_X : z \mapsto T^{|u|}(z)$ for $z \in I_u$ is a IET on the alphabet X.

Theore m [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The family of regular IET is closed under maximal bifix decoding.

Unique ergodicity

In the associated shift space, the map $\mu : [w] \mapsto |I_w|$ is an invariant probability measure.

QUESTION : Is it the only one ?

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Corollary

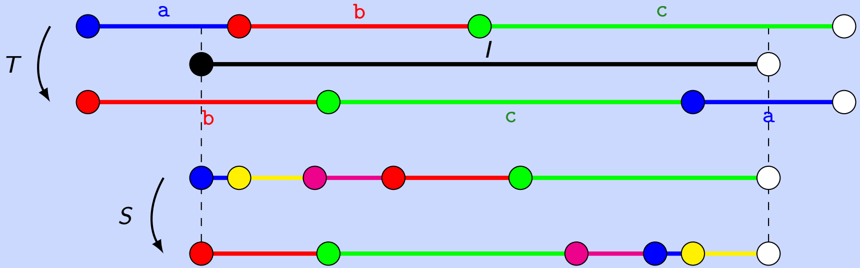
Dendric shift spaces are **not** in general uniquely ergodic (even when minimal).

Induced transformations

Let T be a minimal IET and $I \subset [0,1[$.
The *transformation induced* by T (or *first return map* of T) on I is $S : I \to I$ defined by

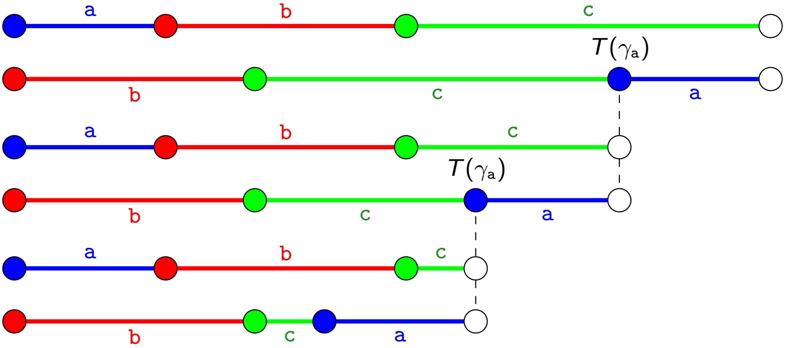
$$S(z) = T^n(z) \qquad \text{with } n = \min\{k > 0 \mid T^k(z) \in I\}$$

Example



Right Rauzy induction

We induce T on $\left[0, \max_{a \in \mathcal{A}}\{\gamma_a, T(\gamma_a)\}\right[$.



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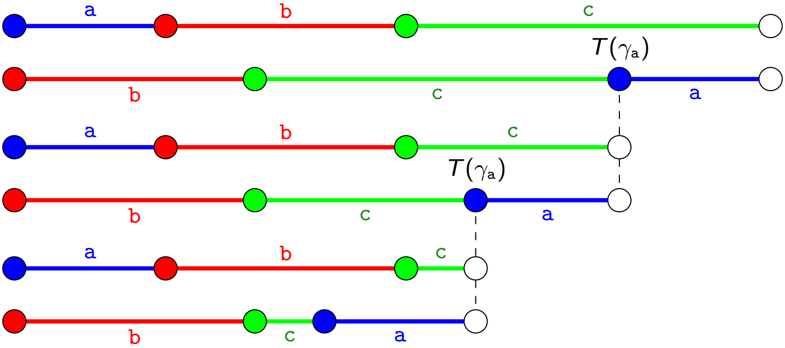
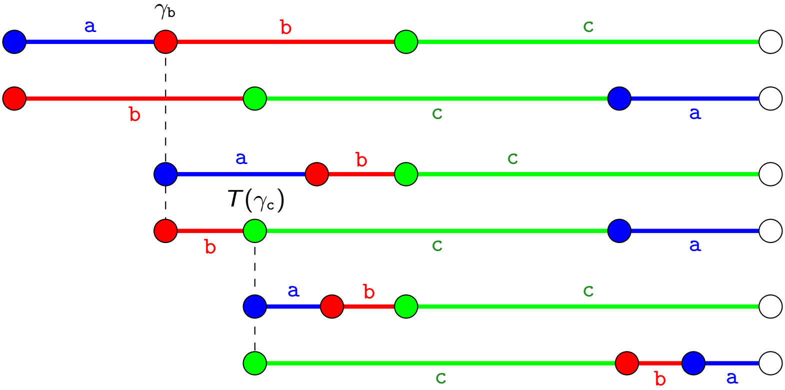


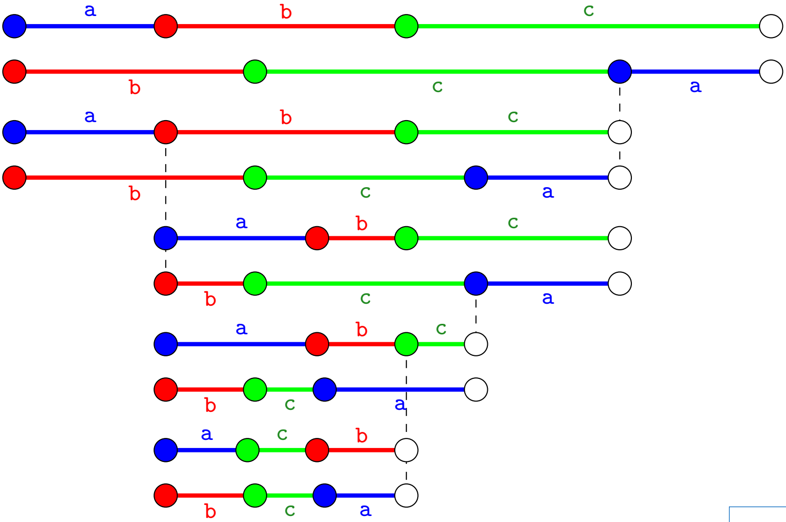
Theorem [Rauzy (1979)]

If T is regular, the right Rauzy induced transformation is regular on the same alphabet.

Left Rauzy induction



Two-sided Rauzy induction



Return words

Theorem [Berthé, De Felice, D., Perrin, Reutenauer, Rindone (2005)]

Let T be regular and $w \in \mathcal{L}(T)$.
The 2-sided induced IET S on J_w (resp., I_w) is regular on the same alphabet.

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Let T be regular and $w \in \mathcal{L}(T)$.
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Moreover, $u \in \mathcal{R}(w)$ if and only if $\Sigma_T(z) = u \cdot \Sigma_T(S(z))$, for some $z \in J_w$.

$$\mathcal{R}(w) = w^{-1} \left(\mathcal{L}(T) \cap \mathcal{A}^+ w \setminus \mathcal{A}^+ w \mathcal{A}^+ \right)$$

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Corollary

The family of regular interval exchange over $\mathrm{Card}\,(\mathcal{A})$ letters is closed under *derivation*.

Corollary (of the Corollary)

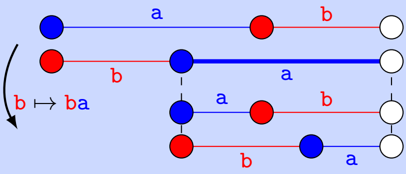
If T is regular, $\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(\mathcal{A})$ for every $w \in \mathcal{L}(T)$.

Return words

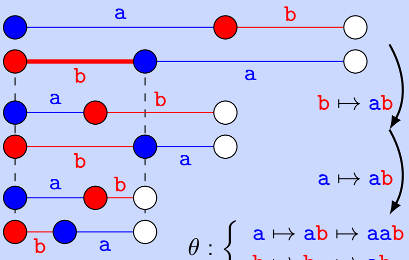
Theorem

Let T regular and S the (2-sided Rauzy) induced IET on I.
There exists $\theta \in \mathrm{Aut}(\mathbb{F}(\mathcal{A}))$ such that $\Sigma_T(z) = \theta(\Sigma_S(z))$ for every $z \in I$

Example (Fibonacci)



$$\theta : \begin{cases} \mathtt{a} \mapsto \mathtt{a} \\ \mathtt{b} \mapsto \mathtt{ab} \end{cases}$$



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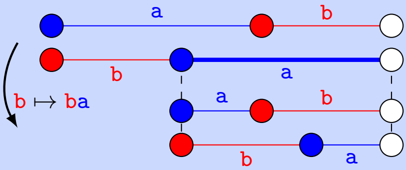

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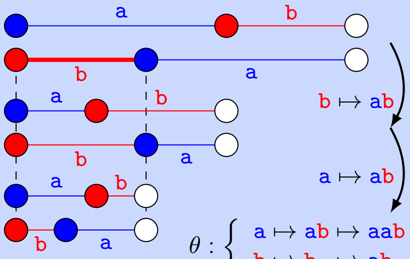
$\mathcal{R}(\mathtt{a}) = \{\mathtt{a}, \mathtt{ba}\}$



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$\mathcal{R}(\mathtt{b}) = \{\mathtt{aab}, \mathtt{ab}\}$

Return words

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Corollary

If T is regular, $\mathcal{R}(w)$ is a basis of $\mathbb{F}(A)$ for every $w \in \mathcal{L}(T)$.

Obrigado pela vossa atenção