Eventually dendric shifts*

Francesco Dolce[1], Dominique Perrin[2]

[1] IRIF, Université Paris Diderot (France),
[2] LIGM, Université Paris Est-Marne-la-Vallée (France)

Abstract

We define a new class of shift spaces which contains a number of classes of interest, like Sturmian shifts used in discrete geometry. We show that this class is closed under conjugacy, a natural transformation obtained by sliding block coding.

1 Introduction

Shift spaces are the sets of two-sided infinite words avoiding the words of a given language F denoted X_F. In this way the traditional hierarchy of classes of languages translates into a hierarchy of shift spaces. The shift space X_F is called of finite type when one starts with a finite language F and sofic when one starts with a regular language F. There is a natural equivalence between shift spaces called conjugacy. Two shift spaces are conjugate if there is a sliding block coding sending bijectively one upon the other (in this case the inverse map has the same form). Many basic questions are still open concerning conjugacy. For example, it is surprisingly not known whether the conjugacy of shifts of finite type is decidable. The complexity of a shift space X is the function $n \mapsto p(n)$ where $p(n)$ is the number of admissible blocks of length n in X. The complexities of conjugate shifts of linear complexity have the same growth rate (see [9, Corollary 5.1.15]).

In this paper, we are interested in shift spaces of at most linear complexity. This class is important for many reasons and includes the class of Stumian shifts which are by definition those of complexity $n + 1$, which play a role as binary codings of discrete lines. Several books are devoted to the study of such shifts (see [9] or [11] for example). We define a new class of shifts of at most linear complexity, called eventually dendric, which includes Sturmian shifts. It extends the class of dendric shifts introduced in [2] (under the name of *tree sets* given to their language) which themselves extend naturally episturmian shifts (also

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called Arnoux-Rauzy shifts) and interval exchange shifts. A *dendric shift* X is defined by introducing the extension graph of a word in the language $\mathcal{L}(X)$ of X and by requiring that this graph is a tree for every word in $\mathcal{L}(X)$. This kind of shifts has many interesting properties which involve free groups. In particular, in a dendric shift X on the alphabet A, the group generated by the set of return words to some word in $\mathcal{L}(X)$ is the free group on the alphabet and, in particular, has $\mathrm{Card}\,(A)$ free generators. This generalizes a property known for Sturmian (and episturmian) shifts whose link with automorphisms of the free group was noted by Arnoux and Rauzy. The class of eventually dendric shifts, introduced in this paper, is defined by the property that the extension graph of every word w in the language of the shift is a tree for every long enough word w.

The paper is organized as follows. In the first section, we introduce the definition of the extension graph and of an eventually dendric shift. In Section 3, we recall some mostly known properties on the complexity of a shift space and of left- or right-special words. We prove a result which characterizes eventually dendric shifts by the extension properties of left-special words (Proposition 4). This result shows us that asymptotically eventually dendric shifts behaves locally in a way similar to Sturmian shifts. In Section 4, we use the classical notion of asymptotic equivalence to give a second characterization of eventually dendric shifts (Proposition 7). In Section 5, we introduce the notion of a simple tree and we prove that for eventually dendric shift, the extension graph of every long enough word is a simple tree (Proposition 8), a property which holds trivially for every word in a Sturmian shift but that is quite surprising for this new larger class of shifts.. Finally, in Section 6 we use the previous results to prove the main result (Theorem 10), namely that the class of eventually dendric shifts is closed under conjugacy. This result shows the robustness of the class of eventually dendric shifts, giving a strong motivation for its introduction.

2 Eventually dendric shifts

Let A be a finite alphabet. We consider the set $A^{\mathbb{Z}}$ of bi-infinite words on A as a topological space for the product topology. The *shift map* $\sigma_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $y = \sigma_A(x)$ if $y_i = x_{i+1}$ for every $i \in \mathbb{Z}$. A *shift space* on the alphabet A is a subset X of the set $A^{\mathbb{Z}}$ which is closed and invariant under the shift, that is such that $\sigma_A(X) = X$ (for more on shift spaces see, for instance, [9]).

We denote by $\mathcal{L}(X)$ the language of a shift space X, which is the set of finite factors of the elements of X. A language $\mathcal{L}$ on the alphabet A is the language of a shift if and only if it is *factorial* (that is contains the factors of its elements) and *extendable* (that is for any $w \in \mathcal{L}$ there are letters $a, b \in A$ such that $awb \in \mathcal{L}$). For $n \geq 0$ we denote $\mathcal{L}_n(X) = \mathcal{L}(X) \cap A^n$ and $\mathcal{L}_{\geq n}(X) = \cup_{m \geq n} \mathcal{L}_m(X)$. For $w \in \mathcal{L}(X)$ and $n \geq 1$, we denote $L_n(w, X) = \{u \in \mathcal{L}_n(X) \mid uw \in \mathcal{L}(X)\}$, $R_n(w, X) = \{v \in \mathcal{L}_n(X) \mid wv \in \mathcal{L}(X)\}$ and $E_n(w, X) = \{(u, v) \in L_n(w, X) \times R_n(w, X) \mid uwv \in \mathcal{L}(X)\}$. The *extension graph* of order n of w, denoted $\mathcal{E}_n(w, X)$, is the undirected graph with set of vertices the disjoint union of $L_n(w, X)$ and $R_n(w, X)$ and with edges the elements of $E_n(w, X)$. When the context is clear, we denote $L_n(w), R_n(w), E_n(w)$ and $\mathcal{E}_n(w)$ instead of $L_n(w, X)$, $R_n(w, X)$, $E_n(w, X)$ and $\mathcal{E}_n(w, X)$ A path in an undirected graph is *reduced* if it does not contain successive equal edges (such a path is also known as *simple*). For any $w \in \mathcal{L}(X)$, since any vertex of $L_n(w)$ is connected to at least one vertex of $R_n(w)$, the bipartite graph $\mathcal{E}_n(w)$ is a tree if and only if there is a unique reduced path between every pair of vertices of $L_n(w)$ (resp. $R_n(w)$).

The shift X is said to be *eventually dendric* with threshold $m \geq 0$ if $\mathcal{E}_1(w)$ is a tree for every word $w \in \mathcal{L}_{\geq m}(X)$. It is said to be *dendric* if we can choose $m = 0$. The languages of dendric shifts were introduced in [2] under the name of tree sets. An important example of dendric shifts is formed by *episturmian shifts* (also called Arnoux-Rauzy shifts), which are by definition such that $\mathcal{L}(X)$ is closed by reversal and such that for every n there exists a unique $w_n \in \mathcal{L}_n(X)$ such that $\mathrm{Card}\,(R_1(w_n)) = \mathrm{Card}\,(A)$ and such that for every $w \in \mathcal{L}_n(X) \setminus \{w_n\}$ one has $\mathrm{Card}\,(R_1(w)) = 1$.

Example 1 Let X be the *Fibonacci shift*, which is generated by the morphism $a \mapsto ab, b \mapsto a$. It is well-known that the Fibonacci shift is a Sturmian shift (see, for example, [9]). The graphs $\mathcal{E}_1(a)$ and $E_3(a)$ are shown in Figure 1.

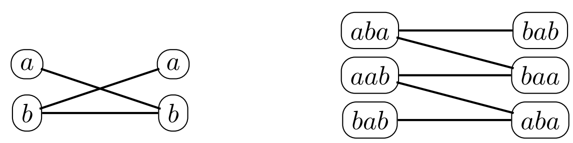

Figure 1: The graphs $\mathcal{E}_1(a)$ (on the left) and $\mathcal{E}_3(a)$ (on the right).

The class of *tree sets of characteristic* $c \geq 1$ introduced in [1, 5] give an example of eventually dendric shifts of threshold 1.

Example 2 Let X be the shift generated by the morphism $a \mapsto ab, b \mapsto cda, c \mapsto cd, d \mapsto abc$ [4]. Its language is a tree set of characteristic 2 (see [1]).

3 Complexity of shift spaces

Let X be a shift space. For a word $w \in \mathcal{L}(X)$, we denote $\ell_k(w) = \mathrm{Card}\,(L_k(w))$, $r_k(w) = \mathrm{Card}\,(R_k(w))$, and $e_k(w) = \mathrm{Card}\,(E_k(w))$. For any $w \in \mathcal{L}(X)$, we have $1 \leq \ell_k(w), r_k(w) \leq e_k(w)$. The word w is *left-k-special* if $\ell_k(w) > 1$, *right-k-special* if $r_k(w) > 1$ and *k-bispecial* if it is both left-k-special and right-k-special. For $k = 1$, we use ℓ, r, e and we simply say special instead of k-special. Given a word w we define the quantity $m(w) = e(w) - \ell(w) - r(w) + 1$. We say that w is *strong* if $m(w) \geq 0$, *weak* if $m(w) \leq 0$ and *neutral* if $m(w) = 0$. It is clear that if $\mathcal{E}_1(w)$ is acyclic (resp. connected, resp. a tree), then w is weak (resp. strong, resp. neutral).

Proposition 1 *Let X be a shift space and let $w \in \mathcal{L}(X)$. If w is neutral, then*

$$\ell(w) - 1 = \sum_{b \in R_1(w)} (\ell(wb) - 1) \tag{1}$$

Set further $p_n(X) = \mathrm{Card}\,(\mathcal{L}_n(X))$, $s_n(X) = p_{n+1}(X) - p_n(X)$ and $b_n(X) = s_{n+1}(X) - s_n(X)$. The sequence $p_n(X)$ is called the *complexity* of X.

The following result is from [3] (see also [2, Lemma 2.12]).

Proposition 2 *We have for all $n \geq 0$,*

$$s_n(X) = \sum_{w \in \mathcal{L}_n(X)} (\ell(w) - 1) = \sum_{w \in \mathcal{L}_n(X)} (r(w) - 1) \quad \text{and} \quad b_n(X) = \sum_{w \in \mathcal{L}_n(X)} m(w).$$

In particular, the number of left-special (resp. right-special) words of length n is bounded by $s_n(X)$.

Proposition 3 *Let X be a shift space. If X is eventually dendric, then the sequence $s_n(X)$ is eventually constant.*

Proof Let N be the threshold of X. By Proposition 2, we have that $b_n(X) = 0$ for all $n \geq N$. Thus $s_{n+1}(X) = s_n(X)$ for all $n \geq N$. ∎

The converse of Proposition 3 is not true, as shown by the following example.

Example 3 Let X be the *Chacon ternary shift*, which is the substitutive shift space generated by the morphism $\varphi : a \mapsto aabc, b \mapsto bc, c \mapsto abc$. It is well known that the complexity of X is $p_n(X) = 2n + 1$ and thus that $s_n = 2$ for all $n \geq 0$ (see [9, Section 5.5.2]). The extension graphs of abc and bca are shown in Figure 2.

Figure 2: The extension graphs $\mathcal{E}_1(abc)$ (on the left) and $\mathcal{E}_1(bca)$ (on the right).

Thus $m(abc) = 1$ and $m(bca) = -1$. Let now α be the map on words defined by $\alpha(x) = abc\varphi(x)$. Let us verify that if the extension graph of x is the graph of Figure 2 on the left, the same holds for the extension graph of $y = \alpha(x)$. Indeed, since $axa \in \mathcal{L}(X)$, the word $\varphi(axa) = aabc\varphi(x)aabc = ayaabc$ is also in $\mathcal{L}(X)$ and thus $(a, a) \in \mathcal{E}_1(y)$. Since $cxa \in \mathcal{L}(X)$ and since a letter c is always preceded by a letter b, we have $bcxa \in \mathcal{L}(X)$. Thus $\varphi(bcxa) = bcyaabc \in \mathcal{L}(X)$ and thus $(c, a) \in \mathcal{E}_1(y)$. The proof of the other cases is similar. The same property holds for a word x with the extension graph on the right of Figure 2. This shows that there is an infinity of words whose extension graph is not a tree and thus the Chacon set is not eventually dendric.

Let X be a shift space. We define $LS_n(X)$ (resp. $LS_{\geq n}(X)$) as the set of left-special words of $\mathcal{L}(X)$ of length n (resp. at least n) and $LS(X) = \bigcup_{n\geq 0} LS_n(X)$.

The following result expresses the fact that eventually dendric shift spaces are characterized by an asymptotic property of left-special words which is a local version of the property defining Sturmian shift spaces.

Proposition 4 *A shift space X is eventually dendric if and only if there is an integer $n \geq 0$ such that any word w of $LS_{\geq n}(X)$ has exactly one right extension $wb \in LS_{\geq n+1}(X)$ with $b \in A$. Moreover, in that case one has $\ell(wb) = \ell(w)$.*

Proof Assume first that X is eventually dendric with threshold m. Then any word w in $LS_{\geq m}(X)$ has at least one right extension in $LS(X)$. Indeed, since $L_1(w)$ has at least two elements and since the graph $\mathcal{E}_1(w)$ is connected, there is at least one element of $R_1(w)$ which is connected by an edge to more than one element of $L_1(w)$. Next, Equation (1) shows that for any $w \in LS_{\geq m}(X)$ which has more than one right extension in $LS(X)$, one has $\ell(wb) < \ell(w)$ for each such extension. Thus the number of words in $LS_{\geq m}(X)$ which are prefix of one another and which have more than one right extension, is bounded by $\mathrm{Card}\,(A)$. This proves that there exists an $n \geq m$ such that for any $w \in LS_{\geq n}(X)$ there is exactly one $b \in A$ such that $wb \in LS(X)$. Moreover, one has then $\ell(wb) = \ell(w)$ by Equation (1).

Conversely, assume that the condition is satisfied for some integer n. For any word w in $\mathcal{L}_{\geq n}(X)$, the graph $\mathcal{E}_1(w)$ is acyclic since all vertices in $R_1(w)$ except at most one have degree 1. Thus w is weak. Let N be the length of w. Then for every word u of length N and every $b \in R_1(u)$, one has $\ell(ub) = 1$ except for one letter b such that $\ell(ub) = \ell(u)$. Thus, by Proposition 2,

$$s_N(X) = \sum_{u\in\mathcal{L}_N(X)} (\ell(u) - 1) = \sum_{v\in\mathcal{L}_{N+1}(X)} (\ell(v) - 1) = s_{N+1}(X).$$

This shows that $b_N = 0$ for every $N \geq n$ and thus, by Proposition 2 again, all words in $\mathcal{L}_{\geq n}(X)$ are neutral. Since all graphs $\mathcal{E}_1(w)$ are moreover acyclic, this forces that these graphs are trees and thus that X is eventually dendric with threshold n. ∎

EXAMPLE 4 Let X be the *Tribonacci shift*, which is the episturmian shift generated by the substitution $\varphi : a \mapsto ab, b \mapsto ac, c \mapsto a$ and let α be the morphism $\alpha : a \mapsto a, b \mapsto a, c \mapsto c$. It can be verified that $\alpha(X)$ satisfies the condition of Proposition 4 with $n = 4$ and thus it is dendric with threshold at most 4. The threshold is actually 4 since $m(a^3) = 1$ in $\alpha(X)$.

4 Asymptotic equivalence

The *orbit* of $x \in A^{\mathbb{Z}}$ is the equivalence class of x under the action of the shift transformation. Thus y is in the orbit of x if there is an $n \in \mathbb{Z}$ such that $x = \sigma^n(y)$. We say that x is a shift of y if they belong to the same orbit.

For $x \in A^{\mathbb{Z}}$, denote $x^- = \cdots x_{-2}x_{-1}$ and $x^+ = x_0x_1\cdots$ and $x = x^- \cdot x^+$. When X is a shift space, we denote X^+ the set of right infinite words u such that $u = x^+$ for some $x \in X$. A right infinite word $u \in A^{\mathbb{N}}$ is a *tail* of the two-sided infinite word $x \in A^{\mathbb{Z}}$ if $u = y^+$ for some shift y of x, that is $u = x_nx_{n+1}\cdots$ for some $n \in \mathbb{Z}$. The *right asymptotic equivalence* on a shift space X is the equivalence defined as follows. Two elements x, y of X are right asymptotically equivalent if there exists two shifts x', y' of x, y such that $x'^+ = y'^+$. In other words, x, y are right asymptotic equivalent if they have a common tail (see Figure 3, where, for simplicity, we suppose $x = x'$ and $y = y'$). The classes of the right asymptotic equivalence not coinciding with only one orbit are called *right asymptotic classes* (they are called *asymptotic components* in [7]).

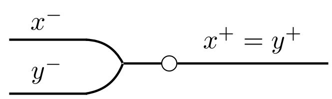

Figure 3: Two right asymptotic sequences x, y.

Example 5 The Fibonacci shift X has only one right asymptotic class. It is formed of the orbits of the two elements $x, y \in X$ such that $x^+ = y^+ = \varphi^\omega(a)$ where $\varphi^\omega(a)$ is the Fibonacci word, that is the right infinite word having all $\varphi^n(a)$ for $n \geq 1$ as prefixes. Indeed, let $x, y \in X$ be such that $x^+ = y^+$ with $x \neq y$. Then all finite prefixes of $x^+ = y^+$ are left-special and thus are prefixes of $\varphi^\omega(a)$ (see, for instance, [9]). Thus $x^+ = y^+ = \varphi^\omega(a)$.

If C is a right asymptotic class, it is a union of orbits. The following result is proved in [7, Lemma 3.2] under a weaker hypothesis that we shall not need here. We give a proof for the sake of completeness.

Proposition 5 *Let X be a shift space such that the sequence $s_n(X)$ is bounded by k. The number of right asymptotic classes is finite and at most equal to k.*

Proof Let $(x_1, y_1), \ldots, (x_h, y_h)$ be h pairs of distinct elements of X belonging to asymptotic classes $C_1, \ldots, C_h$ such that for all $1 \leq i \leq h$ one has $x_i^+ = y_i^+$ and $(x_i)_{-1} \neq (y_i)_{-1}$. For n large enough the prefixes of length n of the x_i^+ are h distinct left-special words and thus $h \leq s_n(X)$ since by Proposition 2 the number of left-special words is bounded by $s_n(X)$. This shows that the number of right asymptotic classes is finite and bounded by k. ∎

Let X be a shift space. For a right asymptotic class C of X, we denote $\omega(C) = \mathrm{Card}\,(o(C)) - 1$ where $o(C)$ is the set of orbits contained in C. For a right infinite word $u \in X^+$, let $\ell_C(u) = \mathrm{Card}\,(a \in A \mid x^+ = au \text{ for some } x \in C)$. We denote by $LS(C)$ the set of right infinite words u such that $\ell_C(u) \geq 2$.

Proposition 6 *Let X be a shift space and let C be a right asymptotic class. Then*

$$\omega(C) = \sum_{u \in LS(C)} (\ell_C(u) - 1) \tag{2}$$

where both sides are simultaneously finite.

In order to prove Proposition 6, we use the following notion. A *cluster of trees* is a directed graph which is the union of a (non-trivial) cycle Γ and a family of disjoint trees (oriented from child to father) T_v with root v indexed by the vertices v on Γ (see Figure 4). It is easy to verify that a finite connected graph is a cluster of trees if and only if every vertex has outdegree 1 and there is a unique strongly connected component. In a cluster of trees, the number of leaves (that is, the leaves of the trees T_v not reduced to their root) is equal to $\sum_u (d^-(u) - 1)$, where d^- stands for the indegree function and the sum runs over the set of internal nodes. Indeed, this is true for one cycle alone since there are no leaves and every internal node has indegree 1. The formula remains valid when suppressing a leaf in one of the trees not reduced to its root.

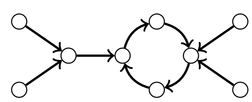

Figure 4: A cluster of trees.

Proof [of Proposition 6] We first suppose that C does not contain periodic points which implies that $LS(C)$ does not contain periodic points either. It is easy to verify that if $u, v \in LS(C)$, there exist $n, m \geq 0$ such that $\sigma^n(u) = \sigma^m(v)$. We build a graph $T(C)$ as follows. The set of vertices of $T(C)$ is $o(C) \cup LS(C)$. There will be for each vertex u of $T(C)$ at most one edge going out of u, called its father.

Let first $x \in C$ and let u be the orbit of x. There is, up to a shift of x, at least one $y \in C$ with $x \neq y$ such that $y^+ = x^+$. Let $n \geq 0$ be the minimal integer such that $x_{-n} \neq y_{-n}$. Then $v = \sigma^{-n+1}(x)^+$ is in $LS(C)$ and depends only on the orbit u of x. We choose the vertex v as the father of u. Next, for every $u \in LS(C)$, we consider the minimal integer, if it exists, such that $v = \sigma^n(u)$ is in $LS(C)$. Then we choose v as the father of u.

Assume now that $\omega(C)$ is finite. Then $LS(C)$ is also finite and $T(C)$ is a finite tree. Indeed, if $u \in LS(C)$, there is at least one $x \in C$ such that $x^+ = u$ and thus such that u is an ancestor of the orbit of x. By the claim made above, any two elements of $LS(C)$ have a common ancestor. Since C does not contain periodic points, two vertices cannot be ancestors of one another. Thus there is a unique element of $LS(C)$ which has no father, namely the unique $u \in LS(C)$ with a maximal number of elements of $o(C)$ as descendants. Since it is an ancestor of all vertices of $T(C)$, this shows that $T(C)$ is a finite tree. Formula (2) now follows from the fact that in any finite tree with n leaves and and a set V of internal vertices, one has $n - 1 = \sum_{v \in V} (d^-(v) - 1)$.

Assume next that the right hand side of Equation (2) is finite. Then the set $LS(C)$ is finite and thus $T(C)$ is again a tree with a finite number of internal nodes. Since the degree of each node is finite, it implies that it has also a finite number of leaves. Thus $\omega(C)$ is finite and Equation (2) also holds.

Finally, assume that C contains a periodic point. It follows from the definition of a right asymptotic class that there is exactly one such periodic orbit, since two periodic points having a common tail are in the same orbit. The proof follows the same lines as in the first case, but this time $T(C)$ will be a cluster of trees instead of a tree. The set of leaves of $T(C)$ is, as above, the set $o(C)$ of non periodic orbits and the the other vertices are the elements of $LS(C)$. The unique father of a vertex is defined in the same way as above. The fact that there is a unique strongly connected component is a consequence of the fact that there is a unique periodic orbit in C. Finally, Formula (2) holds with since the number of leaves is equal to $\sum(d^-(u)-1)-1$, where the sum runs over the set of internal nodes and the -1 corresponds to the unique periodic orbit. ∎

Example 6 Consider again the image $\alpha(X)$ of the Tribonacci shift by the morphism $\alpha : a \mapsto a, b \mapsto a, c \mapsto c$ (Example 4). There is one right asymptotic class C made of three orbits represented in Figure 5 on the left. The class is formed of the orbits of x, y, z where $x^+ = \alpha(\varphi^\omega(a))$ and $y^+ = z^+ = aax^+$. The tree $T(C)$ is shown on the right.

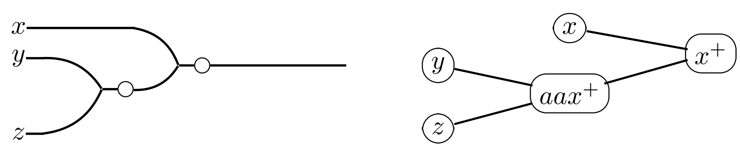

Figure 5: The right asymptotic class C and the tree $T(C)$.

Using Proposition 6 we can give a characterization of eventually dendric shift spaces in terms of right asymptotic classes (for the proof, see [6]). We denote $\omega(X) = \sum \omega(C)$, where the sum is over the right asymptotic classes C of X.

Theorem 7 *A shift space X is eventually dendric if and only if:*

1. *The sequence $s_n(X)$ is eventually constant, and*
2. *We have $\lim s_n(X) = \omega(X)$.*

For example, the Tribonacci shift is such that $s_n(X) = 2$ for every $n \geq 0$ and $\omega(X) = 2$ since there is only one asymptotic class made of 3 orbits. Note that the Chacon shift X satisfies condition 1 of Proposition 7 but not condition 2. Indeed, one can verify that $s_n(X) = 2$ for all $n \geq 0$ but $\omega(X) = 1$.

5 Simple trees

The *diameter* of a tree is the maximal length of simple paths. We call a tree *simple* if its diameter is at most 3. Note that if a simple tree is the extension graph $\mathcal{E}_n(w)$ in some shift space X of a bispecial word w, then the diameter of $\mathcal{E}_n(w)$ is at least 3, and it is exactly 3 if and only if any two vertices of $\mathcal{E}_n(w)$ on the same side (that is, both in $L_n(x)$ or both in $R_n(w)$) are connected to a common vertex on the opposite side. For example, if X is the Fibonacci shift, then $\mathcal{E}_1(a)$ is simple while $\mathcal{E}_3(a)$ is not (see Example 1).

We prove the following additional property of the graphs $\mathcal{E}_k(w)$.

Proposition 8 *Let X be an eventually dendric shift space. For any $k \geq 1$ there exists an $n \geq 1$ such that $\mathcal{E}_k(w)$ is a simple tree for every $w \in \mathcal{L}_{\geq n}(X)$.*

We first prove the following lemma.

Lemma 9 *Let X be an eventually dendric shift space. For every $k \geq 1$ there is an $n \geq 1$ such that if $p, w \in \mathcal{L}(X)$ with $|p| \leq k$ and $|w| \geq n$ are such that $pw, w \in LS(X)$, then pw, w have a unique right extension in $LS(X)$ for some letter $b \in A$ which is moreover such that $\ell(pwb) = \ell(pw)$ and $\ell(wb) = \ell(w)$.*

Proof Consider two right asymptotic classes C, D and let $u \in LS(C)$, $v \in LS(D)$. If C, D are distinct, we cannot have $pu = v$ for some word p. Thus there is an integer n such that if w is the prefix of length n of u, then pw is not a prefix of v. Since there is a finite number of words p of length at most k, a finite number of right asymptotic classes (by Proposition 5) and since for each such class the set $LS(C)$ is finite (by Proposition 6), we infer that for every k there exists an n such that for every pair of right asymptotic classes C, D and any $u \in LS(C), v \in LS(D)$, if w is a prefix of u and pw a prefix of v, with $|p| \leq k$ and $|w| = n$, then $C = D$.

Next, assume that w is a prefix of u and pw a prefix of v with $u, v \in LS(C)$ for some right asymptotic class C. If $v \neq pu$, then there is a right extension w' of w such that pw' is not a prefix of v. By contraposition, if n is large enough, we have $v = pu$. We thus choose n large enough so that: all elements of $LS(C)$ for all right asymptotic components C have distinct prefixes of length n and such that for every pair of asymptotic classes C, D and any $u \in LS(C), v \in LS(D)$, if w is prefix of u and pw is prefix of v with $|p| \leq k$ and $|w| = n$ then $C = D$ and $pu = v$. We moreover assume that n is large enough so that the condition of Proposition 4 holds. Consider p, w with $|p| \leq k$ and $|w| = n$ such that pw, w are left-special. By condition 1, there are right asymptotic components C, D and elements $u \in LS(C)$ and $v \in LS(D)$ such that w is a prefix of u and pw a prefix of v. Because of condition 2, we must have $\sigma^k(v) = u$ (and in particular $C = D$). Thus there is a unique letter $b \in A$ such that $wb, pwb \in LS(X)$ which is moreover such that $\ell(wb) = \ell(w)$ and $\ell(pwb) = \ell(pw)$ by Proposition 4. ■

Proof [of Proposition 8] We choose n such that Proposition 4 and Lemma 9 hold. We prove by induction on h with $1 \leq h \leq k$ that for any $p, q \in L_h(w)$

there is an $r \in R_k(w)$ such that $pwr, qwr \in \mathcal{L}(X)$. This implies that every reduced path in the tree $\mathcal{E}_\ell(w)$ has length at most three, and thus that $\mathcal{E}_\ell(w)$ is a simple tree. The property is true for $h = 1$. Indeed, set $p = a$ and $q = b$. Apply iteratively Proposition 4 to obtain letters $c_1, \ldots, c_k$ such that $\ell(wc_1 \cdots c_i) = \ell(wc_1 \cdots c_i c_{i+1})$ and set $r = c_1 \cdots c_k$. Then $awr, bwr \in \mathcal{L}(X)$.

Assume next that the property is true for $h-1$ and consider $ap, bq \in L_h(w)$ with $a, b \in A$. Replacing if necessary w by some longer word, we may assume that p, q end with different letters and thus that w is left-special. By the induction hypothesis, there is a word $r \in R_k(w)$ such that $pwr, qwr \in \mathcal{L}(X)$. By Lemma 9, the first letter of r is the unique letter c such that $\ell(pwc) = \ell(pw)$ and $\ell(qwc) = \ell(qw)$. Thus $apwc, bqwc \in \mathcal{L}(X)$. Applying Lemma 9 iteratively in this way, we obtain that $apwr, bqwr \in \mathcal{L}(X)$. ■

6 Conjugacy

Let A, B be two alphabets, and $X \subset A^{\mathbb{Z}}$ and $Y \subset B^{Z}$ be two shift spaces. A map $\phi : X \to Y$ is called a *sliding block code* if there exists $m, n \in \mathbb{N}$ and a map $f : \mathcal{L}_{m+n+1}(X) \to B$ such that $\phi(x)_i = f(x_{i-m} \cdots x_{i+n})$ for all $i \in \mathbb{Z}$ and $x = (x_i) \in X$. It can be shown that a map $\phi : X \to Y$ is a sliding block code if and only if it is continuous and commutes with the shift, that is $\phi \circ \sigma_A = \sigma_B \circ \phi$ (see, for instance, [10]). Two shift spaces X, Y are said to be *conjugate* when there is a bijective sliding block code $\phi : X \to Y$. The following result shows that the property of being eventually dendric is a dynamical property, in the sense that it only depends on the class of a shift space under conjugacy.

Theorem 10 *The class of eventually dendric shift spaces is closed under conjugacy.*

We first treat the following particular case of conjugacy. Let X be a shift space on the alphabet A and let $k \geq 1$. Let $f : \mathcal{L}_k(X) \to A_k$ be a bijection from the set $\mathcal{L}_k(X)$ of blocks of length k of X onto an alphabet A_k. The map $\gamma_k : X \to A_k^{\mathbb{Z}}$ defined for $x \in X$ by $y = \gamma_k(x)$ if, for every $n \in \mathbb{Z}$, $y_n = f(x_n \cdots x_{n+k-1})$ is the k-th *higher block code* on X. The shift space $X^{(k)} = \gamma_k(X)$ is called the k-th *higher block shift space* of X. It is well known that the k-th higher block code is a conjugacy. We extend the bijection $f : \mathcal{L}_k(X) \to A_k$ to a map still denoted f from $\mathcal{L}_{\geq k}(X)$ to $\mathcal{L}_{\geq 1}(X^{(k)})$ by $f(a_1 a_2 \cdots a_n) = f(a_1 \cdots a_k) \cdots f(a_{n-k+1} \cdots a_n)$. Note that all nonempty elements of $\mathcal{L}(X^{(k)})$ are image by f of elements of $\mathcal{L}(X)$, that is, $\mathcal{L}(X^{(k)}) = \{f(w) \mid w \in \mathcal{L}_{\geq k}(X)\} \cup \{\varepsilon\}$.

Example 7 Let X be the Fibonacci shift. We show that the 2-block extension $X^{(2)}$ of X is eventually dendric with threshold 1. Set $A_2 = \{u, v, w\}$ with $f : aa \mapsto u, ab \mapsto v, ba \mapsto w$. Since X is dendric, the graph $\mathcal{E}_1(w)$ is a tree for every word $w \in \mathcal{L}(X^{(2)})$ of length at least 1 (but not for $w = \varepsilon$). Thus $X^{(2)}$ is eventually dendric. It is actually a tree shift space of characteristic 2 since the graph $\mathcal{E}_1(\varepsilon)$ is the union of two trees (see Figure 6).

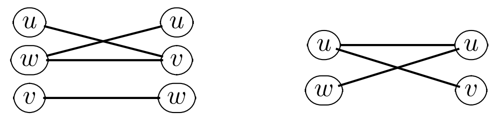

Figure 6: The extension graphs $\mathcal{E}_1(\varepsilon)$ (on the left) and $\mathcal{E}_1(vw)$ (on the right).

Lemma 11 *For every $k \geq 1$, the k-th higher block shift space $X^{(k)}$ is eventually dendric if and only if X is eventually dendric.*

Proof We define for every $w \in \mathcal{L}_{\geq k}(X)$ a map from $\mathcal{E}_1(w)$ to $\mathcal{E}_1(f(w))$ as follows. To every $a \in L_1(w)$, we associate the first letter $\lambda(a)$ of $f(aw)$ and to every $b \in R_1(w)$, we associate the last letter $\rho(b)$ of $f(wb)$. Then, since $f(awb) = \lambda(a)f(w)\rho(b)$, the pair (a, b) is in $E_1(w)$ if and only if $(\lambda(a), \rho(b))$ is in $E_1(f(w))$. Thus, the maps λ, ρ define an isomorphism from $\mathcal{E}_1(w)$ onto $\mathcal{E}_1(f(w))$.

Thus we conclude that X is eventually dendric with threshold m if and only if $X^{(k)}$ is eventually dendric with threshold M with $0 \leq M \leq \max(1, m-k+1)$. ■

Example 8 Let X be the Fibonacci shift. For all $k \geq 2$, $X^{(k)}$ is an eventually dendric shift space with threshold 1.

A morphism $\alpha : A^* \to B^*$ is called *alphabetic* if $\alpha(A) \subseteq B$.

Lemma 12 *Let X be an eventually dendric shift space on the alphabet A and let $\alpha : A^* \to B^*$ be an alphabetic morphism which induces a conjugacy from X onto a shift space Y. Then Y is eventually dendric.*

Proof Since α is invertible, there exists map $f : \mathcal{L}_{2r+1}(Y) \to A$, with $r \geq 0$, such that for $x = (x_k)_{k\in\mathbb{Z}}$ and $y = (y_k)_{k\in\mathbb{Z}}$, one has $y = \alpha(x)$ if and only if for every $k \in \mathbb{Z}$, one has $x_k = f(y_{k-r} \cdots y_{k-1}y_k y_{k+1} \cdots y_{k+r})$. We extend the definition of f to a map from $\mathcal{L}_{\geq 2r+1}$ to A: for $w = b_{1-r} \cdots b_{n+r} \in \mathcal{L}_{\geq 2r+1}(Y)$, set $f(w) = a_1 \cdots a_n$ where $a_i = f(b_{i-r} \cdots b_i \cdots b_{i+r})$. Note that if $u = f(w)$ and $w = svt$ with $s, t \in \mathcal{L}_r(Y)$, then $v = \alpha(u)$. Let n be the integer given by Proposition 8 for $k = r + 1$. We claim that every graph $\mathcal{E}_1(w)$ for $|w| \geq n + 2r$ is a tree. Let indeed $s, t \in \mathcal{L}_r(Y)$ and $v \in \mathcal{L}_{\geq n}(Y)$ be such that $w = svt$. Let $u = f(svt)$. Let $E'_k(u) = \{(p, q) \in L_k(u) \times R_k(u) \mid \alpha(puq) \in BwB\}$ and let $L'_k(u)$ (resp. $R'_k(u)$) be the set of $p \in L_k(u)$ (resp. $q \in R_k(u)$) which are connected to $L_k(u)$ (resp. $R_k(u)$) by an edge in $E'_k(u)$. Let $\mathcal{E}'_k(u)$ be the subgraph of $\mathcal{E}_k(u)$ obtained by restriction to the set of vertices which is the disjoint union of $L'_k(u)$ and $R'_k(u)$ (and that thus has $E'_k(u)$ as set of edges).

The graph $\mathcal{E}'_k(u)$ is a simple tree. Indeed, by Proposition 8, the graph $\mathcal{E}_k(u)$ is a simple tree. We may assume that u is k-bispecial (otherwise, the property is obviously true). Let (p, q) be an edge of $\mathcal{E}'_k(u)$. Then (p, q) is an edge of $\mathcal{E}_k(u)$ and since the latter is a simple tree either p is the unique vertex in $L_k(u)$ such that pu is right-special or q is the unique vertex in $R_k(u)$ such that uq is left-special (both cases can occur simultaneously). Assume the first case, the other being proved in a symmetric way. If (p', q') is another edge of $\mathcal{E}'(u)$, then (p, q') is an edge of $\mathcal{E}_k(u)$. Since $\alpha(p) \in Bs$ and $\alpha(q) \in tB$, we have actually $(p, q') \in E'_k(u)$. Thus $\mathcal{E}'_k(u)$ contains the two vertices of $\mathcal{E}_k(u)$ connected to more than one other vertex and this implies that $\mathcal{E}'_k(u)$ is a simple tree. For $p \in L'_k(u)$, let $\lambda(p)$ be the first letter of $\alpha(p)$ and for $q \in R'_k(u)$, let $\rho(q)$ be the last letter of $\alpha(q)$.

The graph $\mathcal{E}_1(w)$ is the image by the maps λ, ρ of the graph $\mathcal{E}'_k(u)$. Indeed, one has $(a, b) \in E_1(w)$ iff there exist $(p, q) \in E'_k(u)$ such that $\lambda(p) = a$ and $\rho(q) = b$. Let us consider a graph homomorphism ϕ preserving bipartiteness and such such that left vertices are sent to left vertices and right vertices to right ones: Then, it is easy to verify that the image of a simple tree by ϕ is again a simple tree. Thus $\mathcal{E}_1(w)$ is a simple tree, which concludes the proof.

■

Proof [of Theorem 10] Every conjugacy is a composition of a higher block code and an alphabetic morphism (see [10, Proposition 1.5.12]). Thus Theorem 10 is a direct consequence of Lemmas 11 and 12. ■

Example 9 The fact that the image of the Tribonacci shift by the morphism α given in Example 4 is eventually dendric is actually a consequence of Theorem 10. Indeed α is an alphabetic morphism and thus a conjugacy. Images of episturmian shift spaces by non trivial alphabetic morphisms have been investigated in [12].

An interesting open question is whether the class of eventually dendric shifts is closed under taking *factors*, that is, images by a sliding block code not necessarily bijective. It would also be interesting to know whether the conjugacy of effectively given eventually dendric shifts is decidable (the conjugacy of substitutive shifts was recently shown to be decidable [8]).

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