An introduction to dendric sets

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Combinatorial and Algebraic Structure Seminar

Praha, 24. září 2019

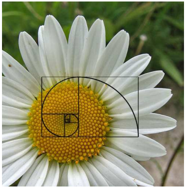
Menu

- Starters
  - Motivation
- Soups
  - Arnoux-Rauzy sets
  - Interval Exchange sets
- Main dishes
  - Extension graphs
  - Dendric and neutral sets
  - Planar dendric sets
  - Maximal bifix decoding
- Side dishes
  - Eventually dendric sets
  - Return and words
- Desserts and coffee
  - Dendric substitutions
  - Stabilizer of dendric words

Fibonači

$$\mathbf{x} = abaababaabaababa\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$



Fibonači

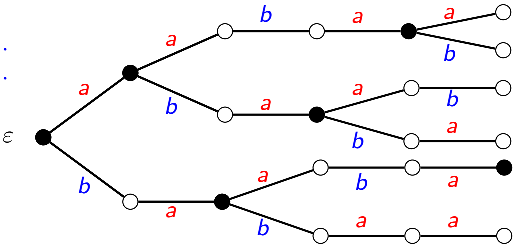
$$\mathbf{x} = abaababaabaababa\cdots$$

The *Fibonači set* (set of factors of $\mathbf{x}$) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonači

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

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$$f : \begin{cases} u & \mapsto & aa \\ v & \mapsto & ab \\ w & \mapsto & ba \end{cases}$$

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$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

$$f^{-1}(x) = \ v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

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Arnoux-Rauzy sets

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An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\text{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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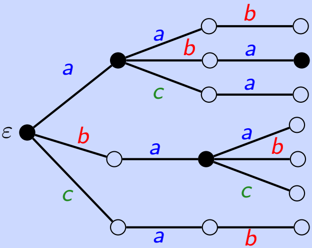
Example (Tribonači)

Factors of the fixed point $\eta^\omega(a)$ of the morphism $\eta : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



$n:$	0	1	2	3	$\cdots$
$p_n:$	1	3	5	7	$\cdots$

$$p_n = 2n + 1$$

2-coded Fibonači

$$f^{-1}(x) = \; v \; u \; w \; w \; v \; u \; w \; w \; \cdots$$

Is the set of factors of $f^{-1}(x)$ an Arnoux-Rauzy set?

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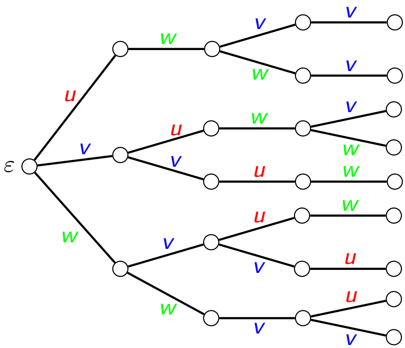

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n :	0	1	2	3	4	···
p_n :	1	3	5	7	9	···

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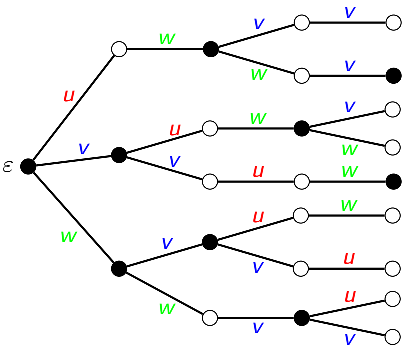
Is the set of factors of $f^{-1}(x)$ an Arnoux-Rauzy set? No!



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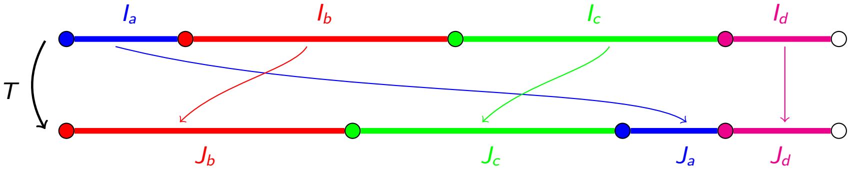
Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T:[0,1[\to[0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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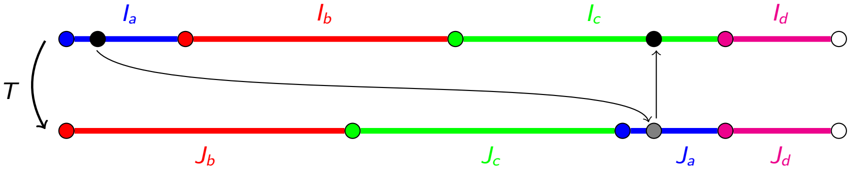

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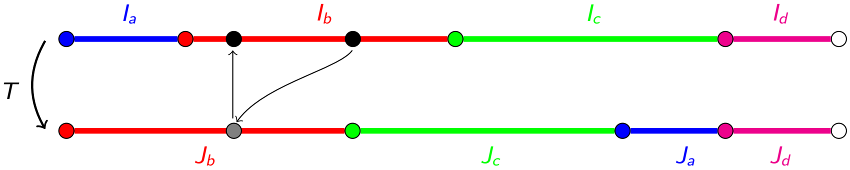

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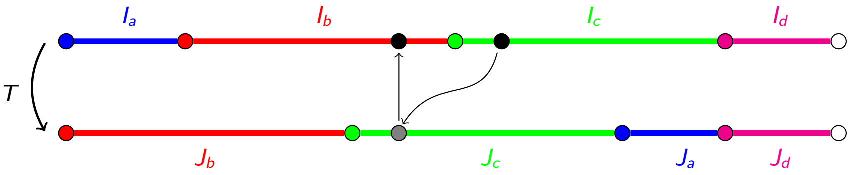

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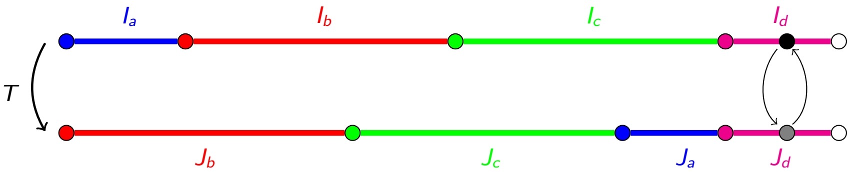

Interval exchanges

T is said to be *minimal* if for any point $z \in [0, 1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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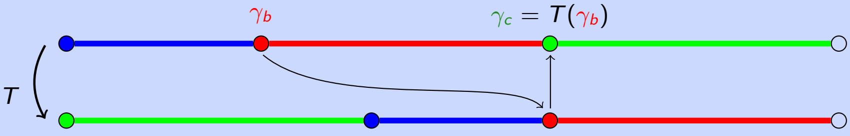
Example (the converse is not true)

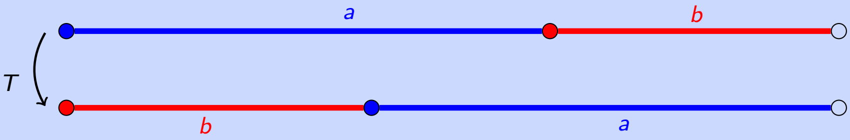


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonači, $z = (3-\sqrt{5})/2$)



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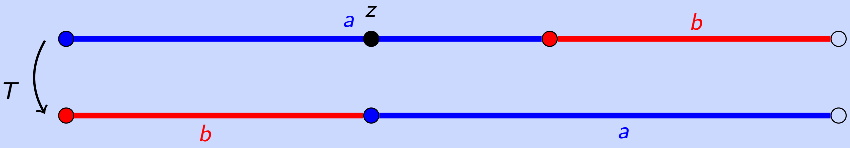

$$\Sigma_T(z) = a$$

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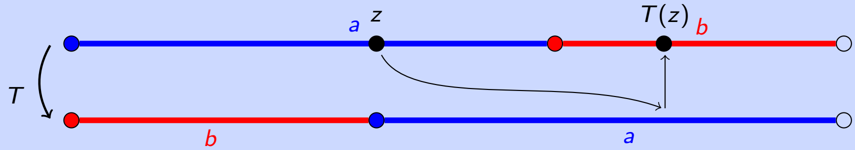

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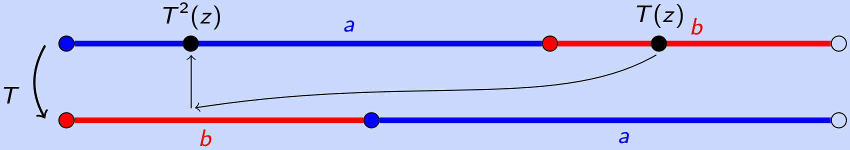

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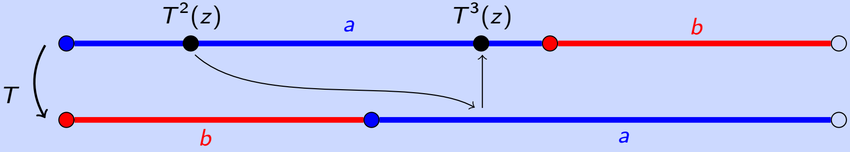

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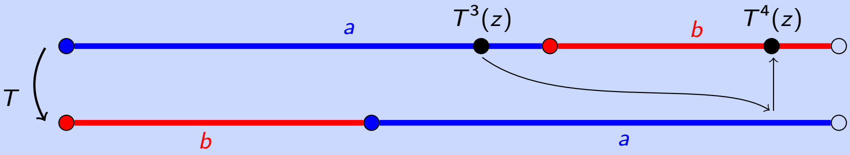

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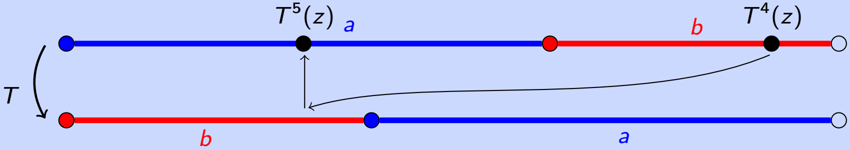


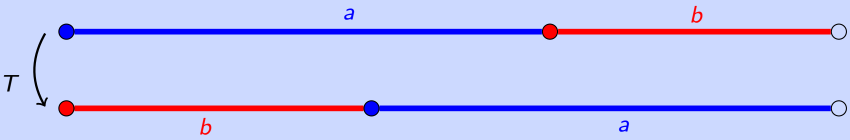
$$\Sigma_T(z) = abaaba \cdots$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonači)



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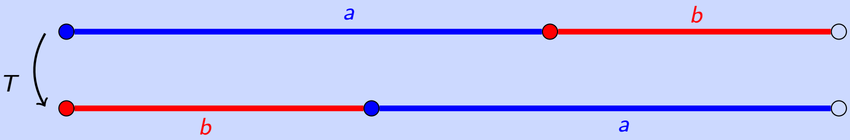

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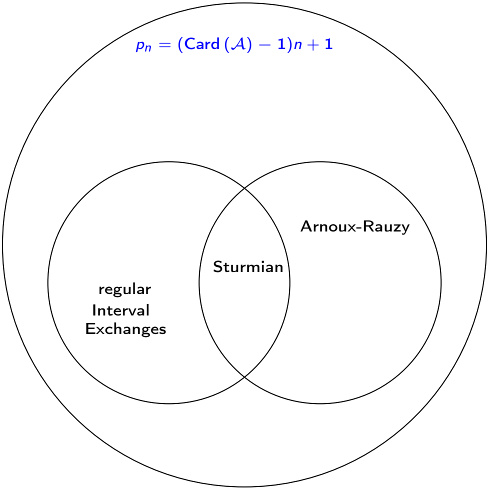
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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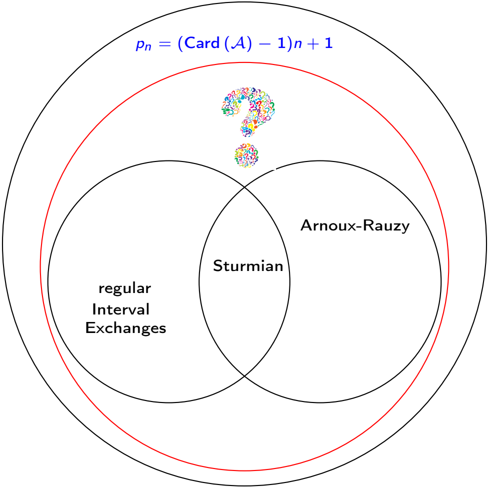

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

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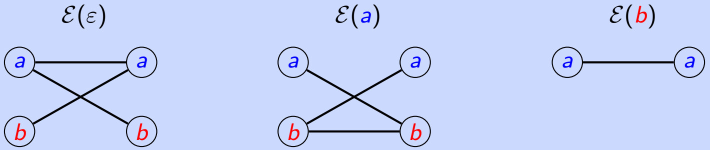

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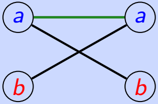
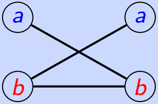


$\mathcal{E}(a)$



$\mathcal{E}(b)$

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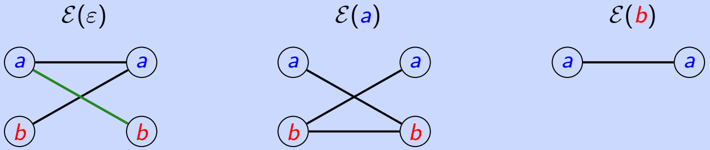

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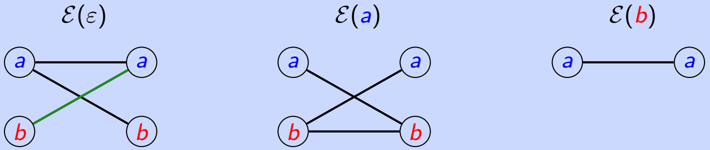

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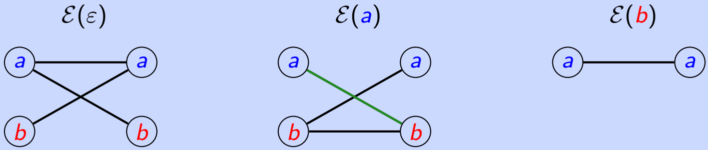

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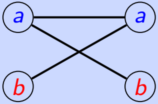

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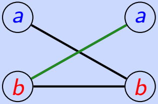

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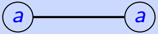

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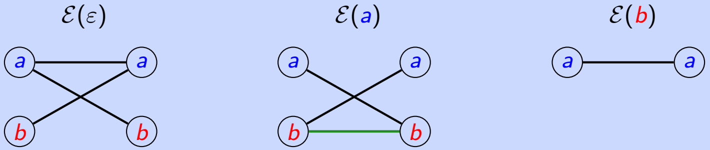

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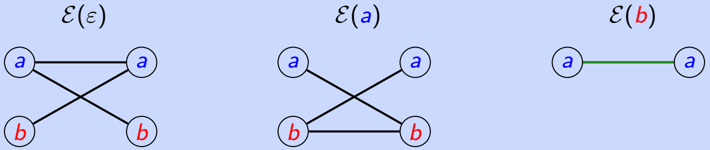


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

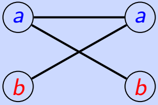
$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

The *multiplicity* of a word w is the quantity

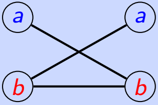
$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

Example (Fibonači, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



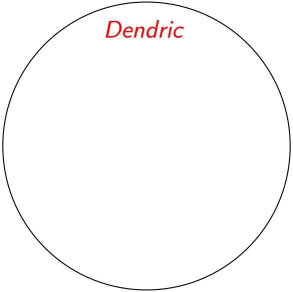
$\mathcal{E}(b)$

$m(a) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral sets

Definition

A language $\mathcal{L}$ is called *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$.



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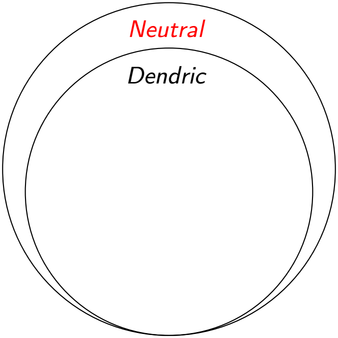

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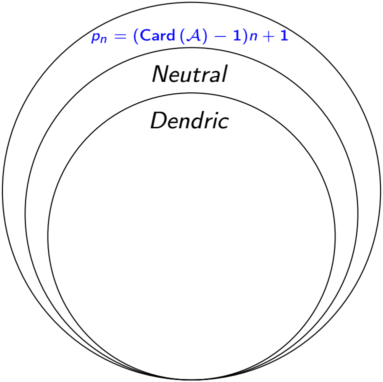

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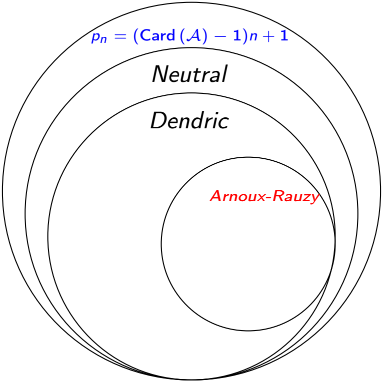

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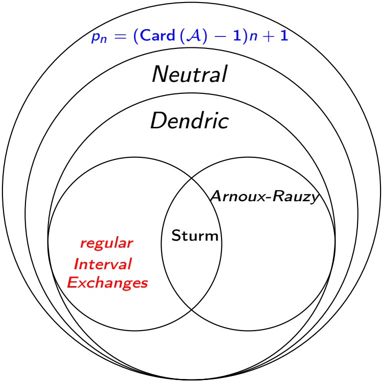
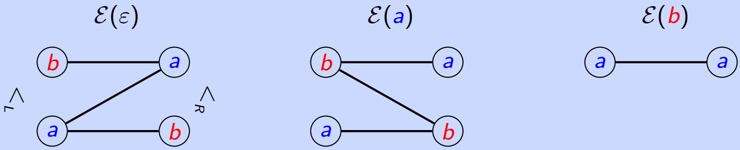


Planar dendric sets

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a set S and a word $w \in S$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_L c \implies b \leq_R d.$$

Example (Fibonači, $b <_L a$ and $a <_R b$)

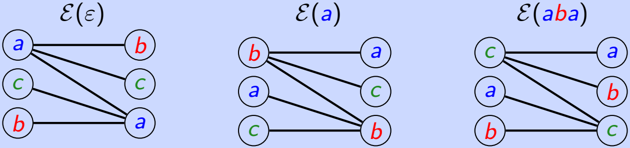


A biextendable set S is a *planar dendric set* w.r.t. $<_L$ and $<_R$ on $\mathcal{A}$ if for any $w \in S$ the graph $\mathcal{E}(w)$ is a tree compatible with $<_L$ and $<_R$.

Planar dendric sets

Example

The *Tribonači set* is **not** a planar dendric set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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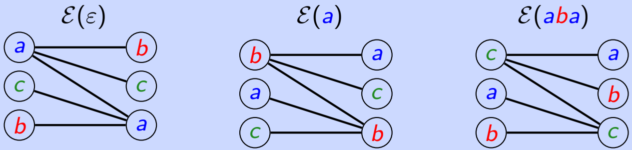
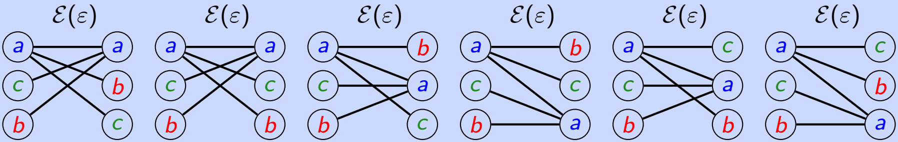


- $a <_L c <_L b$



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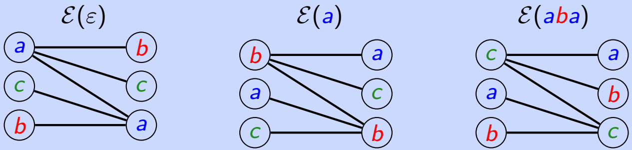
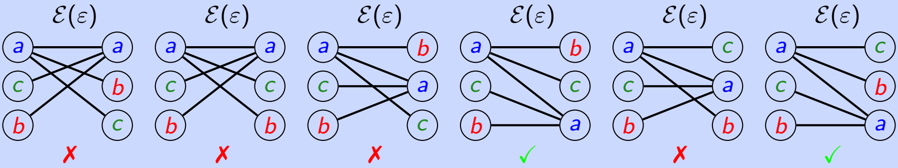


- $\underline{a <_L c <_L b} \implies b <_R c <_R a$ or $c <_R b <_R a$



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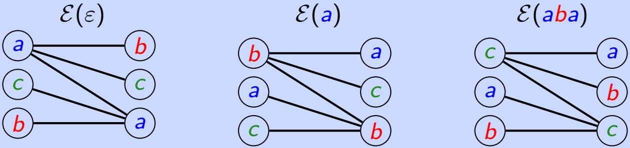

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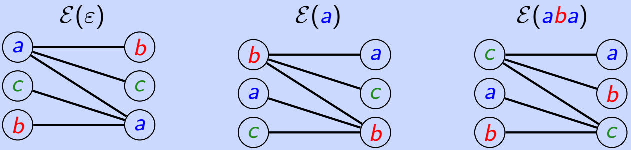

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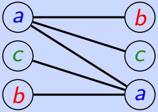
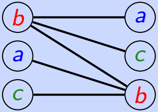
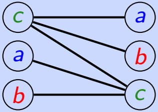


$\mathcal{E}(a)$

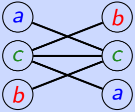


$\mathcal{E}(aba)$



- $\underline{a <_L c <_L b}$ $\Longrightarrow$ $b <_R c <_R a$

$\mathcal{E}(aba)$



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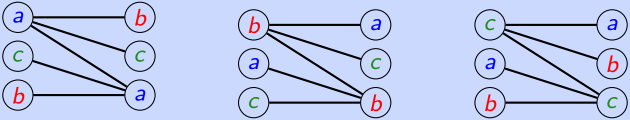
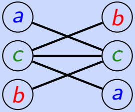
$\mathcal{E}(\varepsilon)$ $\mathcal{E}(a)$ $\mathcal{E}(aba)$



- $a <_L c <_L b \implies$ ↯

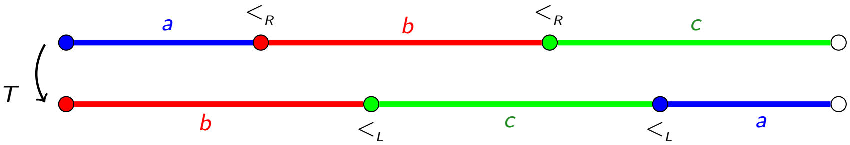
$\mathcal{E}(aba)$

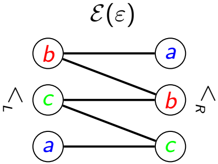


Planar dendric sets

Theorem [Ferenczi, Zamboni (2008)]

A set S is a regular interval exchange set if and only if it is a *recurrent* planar dendric set.





Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u,v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

Example (Fibonači)

$$x = abaababaabaababaababaabaababaaba\cdots$$

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$\mathcal{L}$ is *uniformly recurrent* if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

Example (Fibonači)

$$x = \underbrace{abaa}_{4}\,ba\,\underbrace{baab}_{4}\,\underbrace{aaba}_{4}\,baababaaba\,\underbrace{abab}_{4}\,a\cdots$$

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- Arnoux-Rauzy
- regular Interval Exchanges

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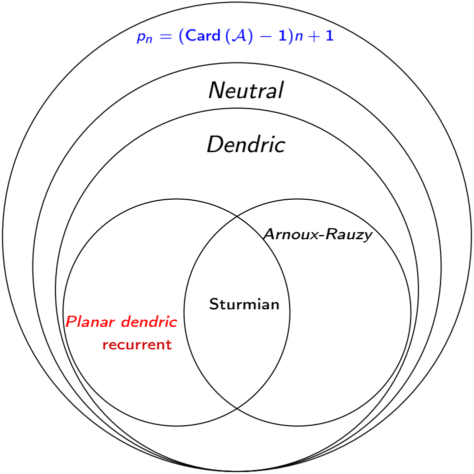
Proposition

Uniform recurrence $\Longrightarrow$ Recurrence.

Dendric and neutral sets



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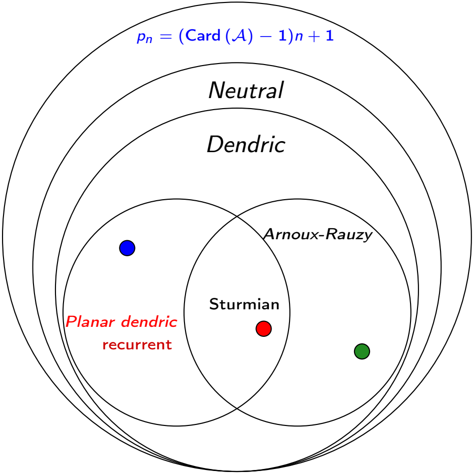
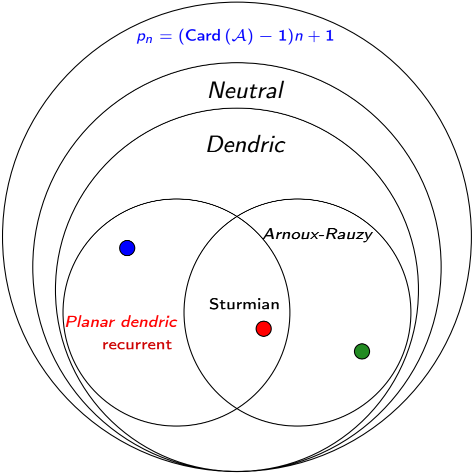


- Fibonači
- Tribonači
- regular IE

Dendric and neutral sets



- • Fibonači
- ? 2-coded Fibonači
- • Tribonači
- ? 2-coded Tribonači
- • regular IE
- ? 2-coded regular IE

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { pivnice, pivo, pivovar }
- ✗ { becherovka, beton, rovka }
- ✗ { s, slivovice, vice }

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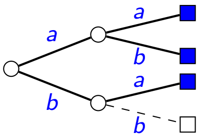
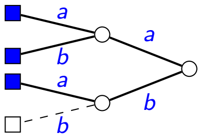
A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

Example (Fibonači)

The set $B = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{bb\}$.





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A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

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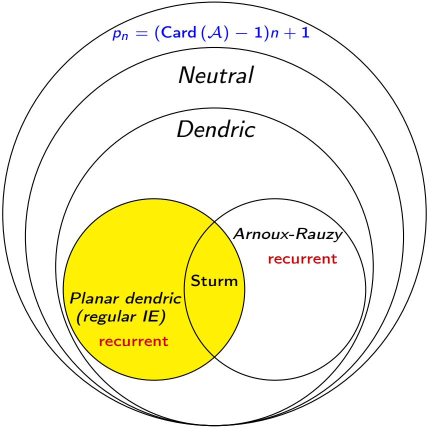
When S is factorial and B is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

The family of *recurrent* planar dendric sets (i.e. regular interval exchange sets) is closed under maximal bifix decoding.



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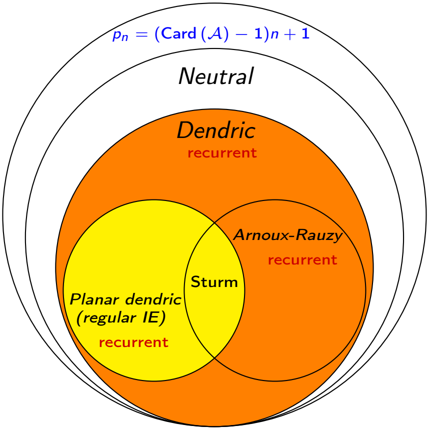
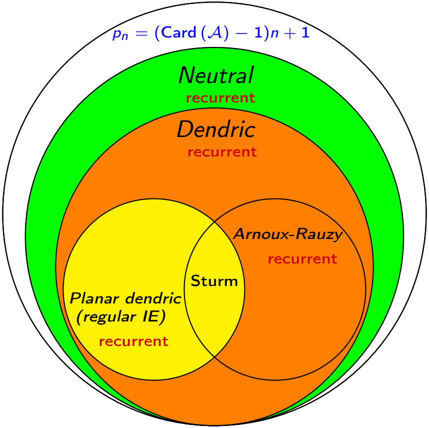


Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016)]

The family of *recurrent* neutral sets is closed under maximal bifix decoding.



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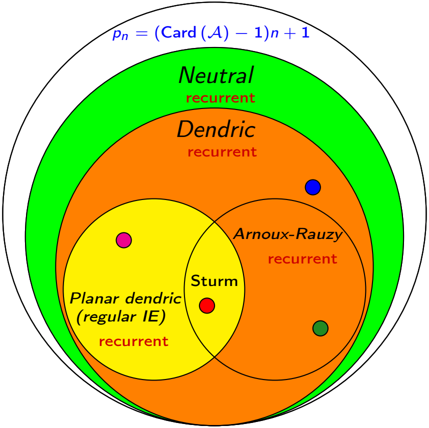

- Fibonači
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A question by Fabien Durand

$$\mathbf{x} = abaababaabaababa \; \cdots$$

A question by Fabien Durand

$$\mathbf{x} = abaababaabaababa \cdots$$

$$\begin{cases} {\color{red}u} & \leftarrow\!\!\dashv & {\color{red}aa} \\ {\color{blue}v} & \leftarrow\!\!\dashv & {\color{blue}ab} \\ {\color{green}w} & \leftarrow\!\!\dashv & {\color{green}ba} \end{cases}$$

A question by Fabien Durand

$$\mathbf{x} = \boxed{ab}\,aababaabaababa \cdots$$

$$\sigma(\mathbf{x}) = v$$

$$\sigma : \begin{cases} u & \leftarrow\!\!\mid & aa \\ v & \leftarrow\!\!\mid & ab \\ w & \leftarrow\!\!\mid & ba \end{cases}$$

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$$\mathbf{x} = abaababaabaababa \cdots$$

$$\sigma(\mathbf{x}) = \color{blue}{v}\color{green}{w}\color{red}{u}\color{blue}{v}\color{green}{w}\color{blue}{v}\color{green}{w}\color{blue}{v}\color{green}{w}\color{red}{u}\color{blue}{v}\cdots$$

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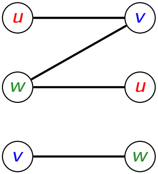
$\mathcal{E}(\varepsilon)$ ✗



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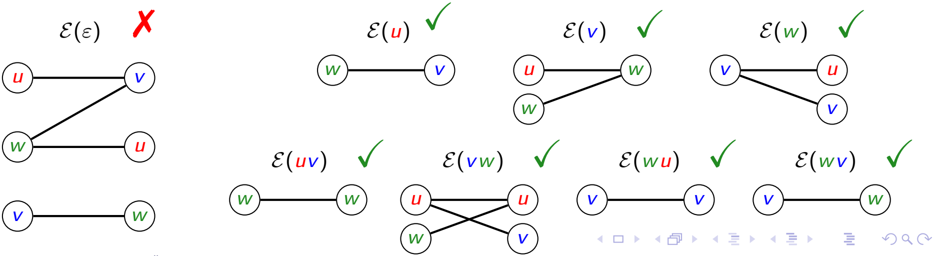

Eventually dendric sets

Definition

A biextendable factorial set S is called a *eventually dendric set* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in S$ s.t. $|w| \geq m$.

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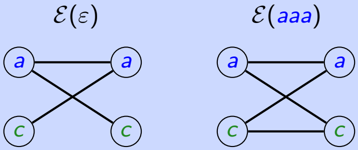
Example (coding of Tribonači)

Let us consider the set $\alpha(S)$, where $\alpha : a, b \mapsto a, \quad c \mapsto c$.



The extension graph of all words of length at least 4 is a tree. (Just trust me!)

Eventually dendric sets

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

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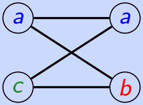
Example (the converse is not true)

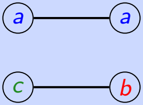
The *Chacon ternary set* is the set of factors of $\varphi^\omega(a)$, where $\varphi : \begin{cases} a \mapsto aabc \\ b \mapsto bc \\ c \mapsto abc \end{cases}$.

One has $p_n = 2n + 1$ ($\Rightarrow s_n = 2$).

$\mathcal{E}(abc)$



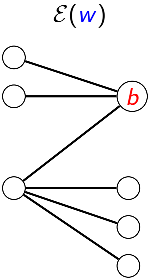
$\mathcal{E}(bca)$



Eventually dendric sets

Theorem [D., Perrin (2019)]

A biextendable factorial set S is eventually dendric if and only if there exists $N \geq 0$ s.t. any left-special word $w \in S$ of length at least N has exactly one right extension wb that is left-special.



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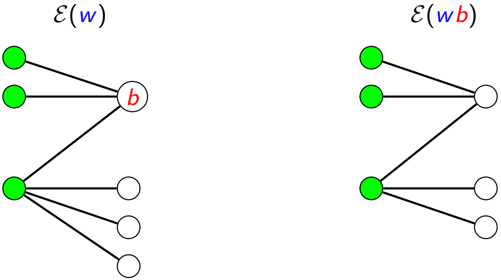
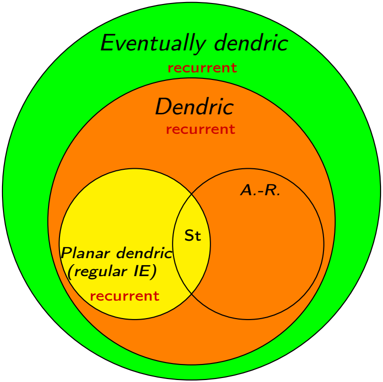
Moreover, in that case one has $L(wb) = L(w)$.



Eventually dendric sets

Theorem [D., Perrin (2019)]

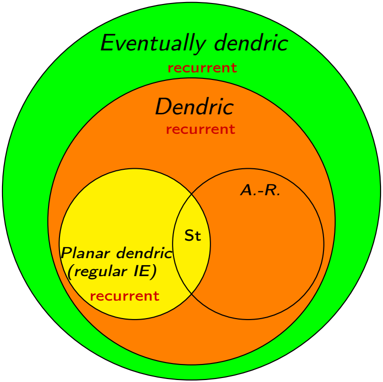
- The family of recurrent eventually dendric sets with threshold m is closed under maximal bifix decoding.



Eventually dendric sets

Theorem [D., Perrin (2019)]

- The family of recurrent eventually dendric sets with threshold m is closed under maximal bifix decoding.
- The family of recurrent eventually dendric sets is closed under conjugacy (the threshold may change).



Return words

A (*left*) *return word* to w in $\mathcal{L}$ is a nonempty word u such that $uw \in \mathcal{L}$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid uw \in \mathcal{L} \cap (wA^+ \setminus A^+wA^+)\}$$

Example (Fibonači)

$\mathcal{R}(0) = \{0, 01\}$

$$\mathbf{x} = 0100101001001010010100100101001001\cdots$$

$\mathcal{R}(00) = \{001, 00101\}$

$$\mathbf{x} = 0100101001001010010100100101001001\cdots$$

Cardinality of return words

Theorem [Vuillon (2001)]

Let $\mathcal{L}$ be a Sturmian set. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = 2.$$

Cardinality of return words



Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008)]

Let $\mathcal{L}$ be a recurrent neutral set. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A)\,.$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008)]

Let $\mathcal{L}$ be a recurrent neutral set. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A)\,.$$

Corollary

A neutral (dendric) set is recurrent if and only if it is uniformly recurrent

Proof. A recurrent set $\mathcal{L}$ is uniformly recurrent if and only if $\mathcal{R}(w)$ is finite for all $w \in \mathcal{L}$.

Return Theorem

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Ridone (2014)]

Let $\mathcal{L}$ be a recurrent dendric set. For every $w \in \mathcal{L}$, $\mathcal{R}(w)$ is a basis of the free group $\mathbb{F}_A$.

Example (Fibonači)

The set $\mathcal{R}(00) = \{001, 00101\}$ is a basis of the free group. Indeed,

$$\begin{aligned} 0 &= 001\,(00101)^{-1}\,001 \\ 1 &= 0^{-1}\,0^{-1}\,001 \end{aligned}$$

Derived sequence
of an infinite word

Let us decode with respect to the return words to the first letter of an infinite word.

Example

$$\mathbf{z} = 0110100110010110100101100110100110010110$$

Derived sequence
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Let us decode with respect to the return words to the first letter of an infinite word.

Example

$\mathcal{R}(0) = \{011, 01, 0\} \quad \longleftrightarrow \quad \{0, 1, 2\}$

$$\mathbf{z} = 0110100110010110100101100110100110010110\cdots$$

$$\mathcal{D}(\mathbf{z}) = 01202101210201202102012101202101210201 21\cdots$$

Derived sequence
of an infinite word

Let us decode with respect to the return words to the first letter of an infinite word.

Example

$\mathcal{R}(0) = \{012, 021, 0121, 02\} \quad \longleftrightarrow \quad \{0, 1, 2, 3\}$

$$\mathbf{z} = 0110100110010110100101100110100110010110\cdots$$

$$\mathcal{D}(\mathbf{z}) = 0120210121020120210201210120210121020121\cdots$$

$$\mathcal{D}^2(\mathbf{z}) = 0123013201232013012301320130123201230132\cdots$$

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\cdots & \quad \cdots
\end{aligned}$$

The sequence $(\mathcal{D}^n(\mathbf{z}))_{n\in\mathbb{N}}$ is called *derived sequence* of $\mathbf{z}$.

Derived sequence

of an infinite word

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REMARK: The alphabets are, in general, different.

Number of derived sequences

Corollary [to the Return Theorem]

For a recurrent *dendric word* $\mathbf{x}$ one has $\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A)$ for any $w \in \mathcal{L}(\mathbf{x})$.
Thus all $\mathcal{D}^n(\mathbf{x})$ are in A^ω.

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Theorem [Durand (1998)]

A uniformly recurrent word $\mathbf{x} \in A^\omega$ is *primitive substitutive* if and only if the set of its derived sequences $\{\mathcal{D}^n(\mathbf{x}) \mid n \in \mathbb{N}\}$ is finite.

Example (Fibonači)

$\text{Card}\,(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}) = 1$, since

$$\mathbf{x} = 0100101001001010010100100101001001 \cdots$$

$$\mathcal{D}(\mathbf{x}) = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1 \cdots$$

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Let $\mathbf{x}$ be a fixed point of a Sturmian substitution $\sigma = \sigma_1\sigma_2\cdots\sigma_q\pi$, with $\sigma_i \in (\mathcal{S}_e \setminus \mathfrak{S}_A)^*$ and $\pi \in \mathfrak{S}_A$ (decomposition in a *normal* form). Then

$$1 \leq \text{Card}\left(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}\right) \leq 3q - 4.$$

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QUESTION: *Can we bound* $\text{Card}\,(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}})$ *when* $\mathbf{x}$ *is recurrent dendric?*

Morphisms and substitutions

A (non-erasing) *morphism* $\sigma : A^* \to B^*$ is a map s.t. $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in A^*$ (and $\sigma(u) \in B^+$ for all $u \in A^+$).

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A substitution is *primitive* if there exists a $k \in \mathbb{N}$ s.t. $b \in \mathcal{L}\left(\sigma^k(a)\right)$ for all $a, b \in A$.

An infinite word of the form $\mathbf{x} = \sigma^\omega(a) = \lim_{n\to\infty} \sigma^n(a)$, with $a \in A$, is a *fixed point* of σ, that is $\sigma(\mathbf{x}) = \mathbf{x}$.

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Proposition

If σ is a primitive substitution, there exists a $k \in \mathbb{N}$ such that σ^k admits a fixed point.

Moreover, all fixed points of σ (or some power of it) have the same language, called the *language of* σ, and this is uniformly recurrent.

Substitutive words

An infinite word $\mathbf{y} \in B^\omega$ is *substitutive* if there exist a substitution σ over B and a morphism $\tau : A^* \to B^*$ such that

$$\mathbf{y} = \tau(\sigma^\omega(b))$$

with $b \in B$. It is said *substitutive primitive* whenever σ is primitive.

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- The map $\varphi : \{0,1\}^* \to \{0,1\}^*$ defined by $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$ is a substitution.

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- The word $\mathbf{x} = \tau(\varphi^\omega(0)) = \tau(0100101001001010\cdots) = \alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\cdots$ is substitutive primitive.

Invertible substitutions

Given an alphabet A, the *free group* $\mathbb{F}_A$ is the set of all words over $A \cup A^{-1}$ which are *reduced* (i.e., $aa^{-1} \equiv a^{-1}a \equiv \varepsilon$ for every $a \in A$).

A substitution $\sigma : A^* \to A^*$ can be extended to a morphism of the free group by defining $\sigma(a^{-1}) = \sigma(a)^{-1}$.

Example

$$
\begin{array}{rrcl}
\varphi: & \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\
 & 0 & \mapsto & 01 \\
 & 1 & \mapsto & 0 \\
 & 0^{-1} & \mapsto & 1^{-1}0^{-1} \\
 & 1^{-1} & \mapsto & 0^{-1}
\end{array}
$$

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A morphism $\sigma : A^* \to A^*$ is *invertible* if its extension $\sigma : \mathbb{F}_A \to \mathbb{F}_A$ is a (positive) automorphism, i.e., if there exists σ^{-1} such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = Id$.

Example

$$\begin{array}{rcl} \varphi : \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\ 0 & \mapsto & 01 \\ 1 & \mapsto & 0 \\ 0^{-1} & \mapsto & 1^{-1}0^{-1} \\ 1^{-1} & \mapsto & 0^{-1} \end{array} \qquad \begin{array}{rcl} \varphi^{-1} : \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\ 0 & \mapsto & 1 \\ 1 & \mapsto & 1^{-1}0 \\ 0^{-1} & \mapsto & 1^{-1} \\ 1^{-1} & \mapsto & 0^{-1}1 \end{array}$$

Tame substitutions

An automorphism σ is *positive* if $\sigma(a) \in A^+$ for every $a \in A$.

An automorphism is *elementary positive* if it is a permutation of A or of the form $\alpha_{a,b}$ or $\widetilde{\alpha}_{a,b}$, with $a, b \in A$ and $a \neq b$, where

$$\alpha_{a,b} : \begin{cases} a \mapsto ab \\ c \mapsto c \qquad \text{if } c \neq a \end{cases} \qquad \text{and} \qquad \widetilde{\alpha}_{a,b} : \begin{cases} a \mapsto ba \\ c \mapsto c \qquad \text{if } c \neq a \end{cases}$$

The set of elementary automorphisms is denoted $\mathcal{S}_e$.
A positive automorphism (resp. substitution) $\sigma \in \mathcal{S}_e^*$ is said to be *tame*.

Example

The set of elementary automorphisms over $A = \{0, 1\}$ is

$$\mathcal{S}_e = \left\{ Id, \pi_{(01)}, \ \alpha_{0,1}, \alpha_{1,0}, \ \widetilde{\alpha}_{0,1}, \widetilde{\alpha}_{1,0} \right\}.$$

The substitution $\varphi = \pi_{(01)}\widetilde{\alpha}_{0,1} : \begin{cases} 0 \mapsto 10 \mapsto 01 \\ 1 \mapsto 1 \ \ \mapsto 0 \end{cases}$ is tame.

Tame and invertible substitutions

tame substitutions $\subset$ invertible substitutions

- Every permutations $\pi \in \mathfrak{S}_A$ is invertible.
- The inverses of

$$\alpha_{a,b} : \begin{cases} a \mapsto ab \\ c \mapsto c & \text{if } c \neq a \end{cases} \quad \text{and} \quad \widetilde{\alpha}_{a,b} : \begin{cases} a \mapsto ba \\ c \mapsto c & \text{if } c \neq a \end{cases}$$

are respectively

$$\alpha_{a,b}^{-1} : \begin{cases} a \mapsto ab^{-1} \\ c \mapsto c & \text{if } c \neq a \end{cases} \quad \text{and} \quad \widetilde{\alpha}_{a,b}^{-1} : \begin{cases} a \mapsto b^{-1}a \\ c \mapsto c & \text{if } c \neq a \end{cases} .$$

Episturmian and epistandard morphisms

epistandard substitutions $\subset$ episturmian substitutions $\subset$ tame substitutions $\subset$ invertible substitutions

The monoid of *episturmian* (or *Arnoux-Rauzy*) *substitutions* is generated by permutations of A and morphisms of the form ψ_a and $\widetilde{\psi}_a$, with $a \in A$, where

$$\psi_a : \begin{cases} a \mapsto a & \\ b \mapsto ab & \text{if } b \neq a \end{cases} \qquad \text{and} \qquad \widetilde{\psi}_a : \begin{cases} a \mapsto a & \\ b \mapsto ba & \text{if } b \neq a \end{cases}$$

The monoid of *epistandard substitutions* is generated by permutations of A and morphisms of the form ψ_a, with $a \in A$ (i.e., no $\widetilde{\psi}_b$).

Episturmian and epistandard morphisms

epistandard substitutions	$\subset$	episturmian substitutions	$\subset$	tame substitutions	$\subset$	invertible substitutions

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Example (Fibonači and Tribonači)

- The substitution $\varphi = \psi_0\, \pi_{(01)} : \begin{cases} 0 \mapsto 1 \mapsto 01 \\ 1 \mapsto 0 \mapsto 0 \end{cases}$ is epistandard.
- The substitution $\eta = \psi_0\, \pi_{(012)} : \begin{cases} 0 \mapsto 1 \mapsto 01 \\ 1 \mapsto 2 \mapsto 02 \\ 2 \mapsto 0 \mapsto 0 \end{cases}$ is epistandard.

Episturmian and epistandard morphisms

epistandard substitutions = episturmian substitutions = tame substitutions = invertible substitutions

$A = \{0, 1\}$

Theorem [Mignosi, Séébold (1993) ; Wen, Wen (1994)]

In the binary case (*Sturmian substitutions*) the four monoids coincide.

Proof. (of the first two equalities)

- For every $a \in \{0, 1\}$, one has $\widetilde{\psi}_a = \pi_{(0,1)}\, \psi_a\, \pi_{(0,1)}$.
- $\alpha_{0,1} = \pi_{(0,1)}\, \psi_0$, $\quad \alpha_{1,0} = \pi_{(0,1)}\, \psi_1\, \pi_{(0,1)}$, $\quad \widetilde{\alpha}_{0,1} = \psi_1$, $\quad \widetilde{\alpha}_{1,0} = \psi_0$.

Epistumian and epistandard morphisms

epistandard substitutions	=	episturmian substitutions	=	tame substitutions	=	invertible substitutions

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Corollary

The monoid of positive automorphisms over a binary alphabet is finitely generated.

Episturmian and epistandard morphisms

epistandard substitutions	$\subsetneq$	episturmian substitutions	$\subsetneq$	tame substitutions	$\subsetneq$	invertible substitutions

Card $(A) \geq 3$

Theorem [Wen, Zhang (1999) ; Richomme (2003)]

The monoid of invertible substitutions over a ternary alphabet is not finitely generated.

Fixed point of substitutions

Theorem

Every Sturmian substitution generates a Sturmian word.

Example

The substitution φ generates the *Fibonači word*

$$\varphi^{\omega}(0) = 0100101001001010\cdots$$

which is Sturmian.

Fixed point of substitutions

Theorem

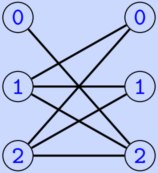
Every Sturmian substitution generates a Sturmian word.

<u>But</u> not every tame substitution admits as a fixed point a dendric word.

Example

$\xi = \alpha_{0,2}\,\alpha_{2,1}\,\alpha_{1,0} : \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 102 \\ 2 \mapsto 21 \end{cases}$ is tame but its fixed point $\xi^\omega(0)$ is not a dendric word.

$\mathcal{E}(\varepsilon)$



Question: *Can we characterize among substitutive dendric words the fixed points?*

Stabilizer

The *stabilizer* of an infinite word $\mathbf{x} \in A^\omega$ is the submonoid of substitutions

$$\mathrm{Stab}(\mathbf{x}) = \{\sigma : A^* \to A^* \mid \sigma(\mathbf{x}) = \mathbf{x}\}$$

A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

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A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

Theorem [Séébold (1998)]

Words generated by Sturmian substitutions are rigid.

Example (Fibonači)

The stabilizer of the Fibonači word $\mathbf{x}$ is $\mathrm{Stab}(\mathbf{x}) = \{\varphi^i \mid i \in \mathbb{N}\}$.

Stabilizer

The *stabilizer* of an infinite word $\mathbf{x} \in A^\omega$ is the submonoid of substitutions

$$\mathrm{Stab}(\mathbf{x}) = \{\sigma : A^* \to A^* \mid \sigma(\mathbf{x}) = \mathbf{x}\}$$

A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

Theorem [Séébold (1998)]

Words generated by Sturmian substitutions are rigid.

Theorem [Krieger (2008)]

Fixed points of strict epistandard morphisms are rigid.

Example (Tribonači)

The stabilizer of the Tribonači word $\mathbf{y} = \eta^\omega(0)$ is $\mathrm{Stab}(\mathbf{y}) = \{\eta^i \mid i \in \mathbb{N}\}$.

Stabilizers of dendric words

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Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

Let $\mathbf{x}$ be a dendric word and $\sigma, \tau \in \mathrm{Stab}(\mathbf{x})$ primitive substitutions.
Then, there exist $i, j \geq 1$ such that $\sigma^i = \tau^j$.

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Let $\mathbf{x}$ be a recurrent dendric word.
There exists a *primitive tame* substitution θ such that for any primitive $\sigma \in \mathsf{Stab}(\mathbf{x})$, one can find a *positive tame automorphism* τ and integers $i, j \geq 1$ such that $\sigma^i = \tau\theta^j\tau^{-1}$.

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If $\mathbf{x}$ is a recurrent dendric word, then any primitive $\sigma \in \mathrm{Stab}(\mathbf{x})$ is invertible (and thus tame).

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QUESTION: *Is any non-trivial element of* $\mathrm{Stab}(\mathbf{x})$ *primitive when* $\mathbf{x}$ *is recurrent dendric?*

Děkuji