Playing with games and words

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Nim Game

(Czech version)

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Initial position: Three piles of beers with arbitrary sizes.
Rules:

i) At each turn a player drinks a positive number of beers from one pile.

Winner: Who drinks the last beer.

Nim Game

Using some math

Denote by (a, b, c) be a game position (a beers on the first pile, etc.).
A position is in $\mathcal{P}$ if there exists a *winning strategy* for the player who plays next.
Otherwise it is in $\mathcal{N}$.

Formally

- $(0, 0, 0) \in \mathcal{P}$;
- $(a, b, c) \in \mathcal{P} \quad \Rightarrow \quad Nim(a, b, c) \subseteq \mathcal{N}$;
- $(a, b, c) \in \mathcal{N} \quad \Rightarrow \quad Nim(a, b, c) \cap \mathcal{P} \neq \emptyset$.

The set $\mathcal{P}$ contains: $(0, 0, 0), (0, 1, 1), (1, 2, 3), (7, 8, 15), \ldots$
The set $\mathcal{N}$ contains: $(0, 0, 1), (1, 2, 2), (2, 3, 4), (8, 11, 17), \ldots$

Nim Game

Using more math

Question: How to determine whether a position is in $\mathcal{P}$?

Theorem [C. Bouton (1904)]

A position (a, b, c) is in $\mathcal{P}$ if its *Nim-sum* is 0.

$$2 \oplus 4 \oplus 6 = 0$$

$$\begin{array}{ccc} & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}$$

$$3 \oplus 6 \oplus 8 \neq 0$$

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 \end{array}$$

Wythoff's Game

A modification of Nim Game

Initial position: Two piles of beers with arbitrary sizes.
Rules: At each turn a player drinks either

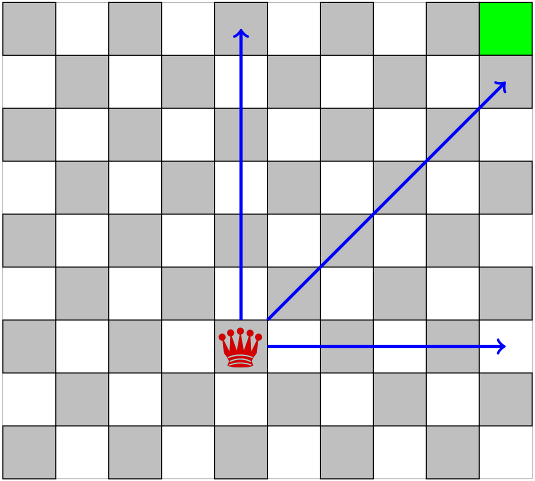
i) a positive number of beers from one pile, or

ii) a positive equal number of beers from both piles.

Winner: Who drinks the last beer.

Wythoff's Game

Playing chess



Wythoff's Game
Safe positions

Question: How to compute the set $\mathcal{P}$?

- $(0,0) \in \mathcal{P}$ but $(n,n) \in \mathcal{N}$ for every $n > 0$;
- if $(a,b) \in \mathcal{P}$ then $(a+k, b+k) \in \mathcal{N}$ for every $k > 0$;

Wythoff's Game

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Theorem [W. Wythoff (1907)]

The set $\mathcal{P}$ is defined by the positions $\{(a_n, b_n)\}_{n\in\mathbb{N}}$, where $(a_0, b_0) = (0,0)$ and

$$\begin{cases} a_n = \mathrm{Mex}\left(\{a_i, b_i \mid 0 \leq i < n\}\right), \\ b_n = a_n + n. \end{cases}$$

Thus $\mathcal{P}$ contains: $(0,0)$, $(1,2)$, $(3,5)$, $(4,7)$, $(6,10)$, $\ldots$.

Wythoff's Game

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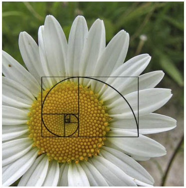
Thus $\mathcal{P}$ contains: $(0,0)$, $(1,2)$, $(3,5)$, $(4,7)$, $(6,10)$, $\ldots$.

Question: Is there another way to compute the set?

Fibonacci word

$$\mathbf{f} = \mathtt{abaababaabaababaaba}\cdots$$

$$\mathbf{f} = \lim_{n\to\infty} \varphi^n(\mathtt{a}) \qquad \text{where} \qquad \varphi : \begin{cases} \mathtt{a} \mapsto \mathtt{ab} \\ \mathtt{b} \mapsto \mathtt{a} \end{cases}$$



Fibonacci word

$$\mathbf{f} = \texttt{abaababaabaababaaba}\cdots$$

Let a_n denote the n^{th} occurrence of a and b_n denote the n^{th} occurrence of b.

$$(a_n)_{n\geq 1} = 1, 3, 4, 6, 8, 9, \ldots \qquad (b_n)_{n\geq 1} = 2, 5, 7, 10, 13, 15, \ldots$$

Theorem [Duchêne, Rigo (2008)]

Let $a_0 = b_0 = 0$. The sequence $(a_n, b_n)_{n\in\mathbb{N}}$ is the Wythoff's sequence.

Proof.

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Proof.

→ All b are created by a, the gaps are filled with a and $a_n = \mathrm{Mex}(\{a_i, b_i \mid 0 \leq i < n\})$.
→ Since **f** starts with ab, then $b_1 = 2 = a_1 + 1$;
Let us suppose that $b_{n-1} = a_{n-1} + n - 1$.

- Since $\varphi(\texttt{aa}) = \texttt{abab}$, if $a_n - a_{n-1} = 1$ then $b_n - b_{n-1} = 2$;
- Since $\varphi(\texttt{aba}) = \texttt{abaab}$, if $a_n - a_{n-1} = 2$ then $b_n - b_{n-1} = 3$;

In both case $b_n = a_n + n$.

Sturmian words

Definition

An infinite word $\mathbf{w}$ is *Sturmian* if it has $n+1$ distinct factors of length n for every $n \geq 0$.

Example (Fibonacci)

$$\mathbf{f} = \mathtt{abaababaabaababa}\cdots$$

$$\mathcal{L}(\mathbf{f}) = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}}_{2}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}}_{3}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}}_{4}, \underbrace{\mathtt{aaba}, \mathtt{abaa}, \mathtt{abab}, \mathtt{baab}, \mathtt{baba}}_{5}, \ldots\}$$

Sturmian words

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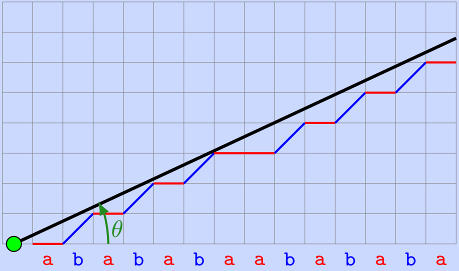
An infinite word $\mathbf{w}$ is *Sturmian* if it has $n+1$ distinct factors of length n for every $n \geq 0$.
A Sturmian word can also be represented geometrically.

Example (Fibonacci)



$$\begin{cases} \mathtt{a} \text{ if } \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 0 \\ \mathtt{b} \text{ if } \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 1 \end{cases}$$

Theorem [W. Wythoff (1907)]

The set $\mathcal{P}$ is defined by the positions $\{(a_n, b_n)\}_{n\in\mathbb{N}}$, where

$$a_n = \lfloor n\tau \rfloor \qquad b_n = \lfloor n\tau^2 \rfloor$$

where $\tau = \frac{1+\sqrt{5}}{2}$ (and thus $\tau^2 = \frac{3+\sqrt{5}}{2}$).

Wythoff's Game

Algebraic characterisation

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Proof.

$\rightarrow$ Easy to see that $b_n - a_n = n$.

$\rightarrow$ Prove that every positive integer appears exactly once is a bit more complicated...

- For every irrational α the set of infinite pairs $\{\lfloor n\alpha \rfloor, \lfloor n\frac{\alpha}{\alpha-1} \rfloor\}_{n\in\mathbb{N}}$ covers $\mathbb{Z}$.
- $\alpha - \frac{\alpha}{\alpha-1} = 1 \quad \Leftrightarrow \quad \alpha = \tau$

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The *golden ration* τ is exactly the *frequence* of a in **f** (and τ^2 the frequence of b).

Modified Wythoff's Games?

always with two piles

Question: Let $\mathbf{x}$ be a Sturmian word. Is it possible to define a new game (*similar rules as Wythoff's one*) such that $(A, B) \in \mathcal{P}$ if and only if $A = a_n$ and $B = b_n$ with a_n (resp. b_n) the n^{th} occurrence of a (resp. of b) in $\mathbf{x}$?

Arnoux-Rauzy words

Definition

An infinite word $\mathbf{w}$ over an alphabet of k letters is an *Arnoux-Rauzy word* if

1. it has $(k-1)n+1$ distinct factors of length n for every $n \geq 0$;
2. its set of factors is closed under reversal and
3. for each lenght only one factor is right special.

Example (Tribonacci: ψ : a $\mapsto$ ab, b $\mapsto$ ac, c $\mapsto$ a)

$$\mathbf{t} = \mathtt{abacabaabacababacabaabaca}\cdots$$

$$\mathcal{L}(\mathbf{t}) = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}, \mathtt{c}}_{3}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{ca}}_{5}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{aca}, \mathtt{baa}, \mathtt{bab}, \mathtt{bac}, \mathtt{cab}}_{7}, \ldots\}$$

Modified Wythoff's Game

Tribonacci game

Initial position: Two piles of beers with arbitrary sizes.
Rules: At each turn a player drinks either

i) a positive number of beers from one pile; or

ii) a positive number α, β and γ of beers from the first, second and third pile whenever $2\max\{\alpha,\beta,\gamma\} \leq \alpha+\beta+\gamma$; or

iii) the same positive number α of beers from two piles and β from the other pile whenever $\beta > 2\alpha > 0$ and $a' < c' < b'$, with (a,b,c) the original position and (a',b',c') the new one.

Winner: Who drinks the last beer.

Tribonacci game and Tribonacci word

$$\mathbf{t} = \mathtt{abacabaabacababacabaabaca}\cdots$$

Let a_n, b_n and c_n denote the n^{th} occurrences of a, b and c in **t** respectively.

$$(a_n)_n = 1, 3, 4, 7, 8, \ldots \quad (b_n)_n = 2, 6, 9, 13, 15, \ldots \quad (c_n)_n = 4, 11, 17, 24, 28, \ldots$$

Theorem [Duchêne, Rigo (2008)]

The set $\{(a_n, b_n, c_n) \mid n \geq 1\}$ is set of $\mathcal{P}$-positions of the Tribonacci game.

Proof. (idea)

$$\begin{cases} a_n = \mathrm{Mex}\left(\{a_i, b_i, c_i \mid 0 \leq i < n\}\right), \\ b_n = a_n + \mathrm{Mex}\left(b_i - a_i,\ c_i - b_i \,|\, 0 \leq i < n\right). \\ c_n = a_n + b_n + n \end{cases}$$

Modified Whythoff's Games?

on two or more piles

Question: Let $\mathbf{x}$ be an Arnaux-Rauzy word. Is it possible to define a new game (*similar rules as Whytoff's one*) such that $(A, B, C) \in \mathcal{P}$ if and only if $A = a_n$,$B = b_n$ and $C = c_n$ with a_n (resp. b_n, c_n) the n^{th} occurrence of a (resp. b, c) in $\mathbf{x}$?

Děkuji za pozornost!

(řekli Alice a Bob)