Journey to the Words

or IETs: from Symbolic Dynamics to Combinatorics on Words

DOLCE Francesco (杜嘉法)

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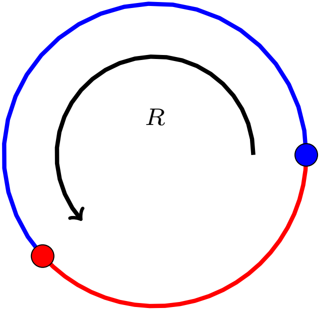
北京航空航天大学, 杭州 – 2025 年 10 月 22 日

Rotations



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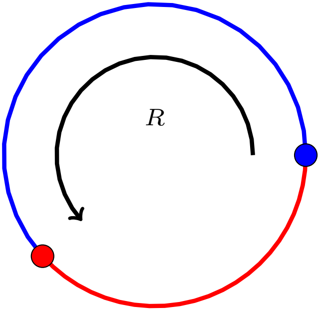

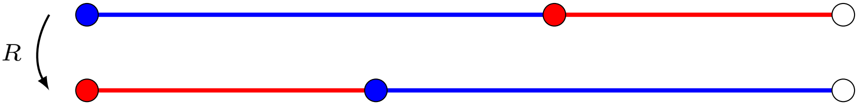

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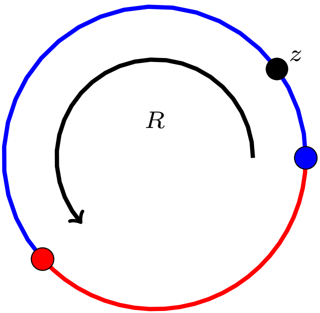

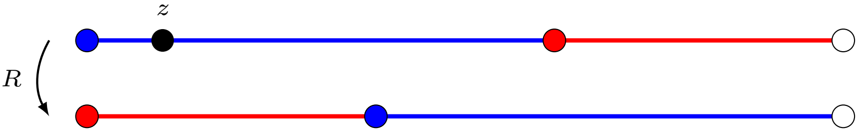

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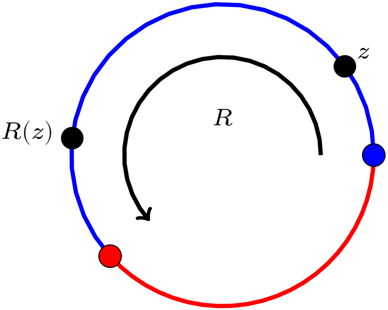

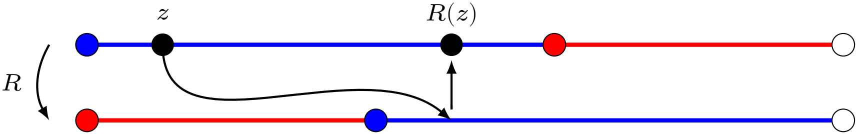

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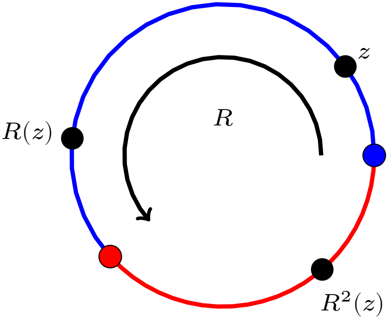

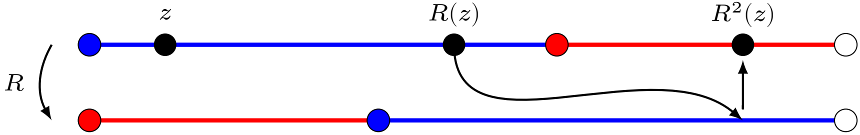

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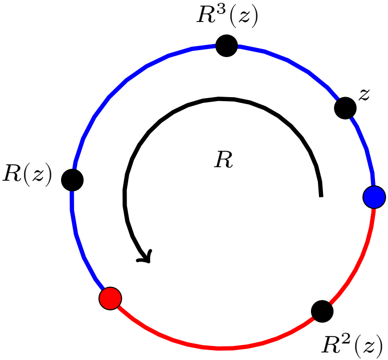
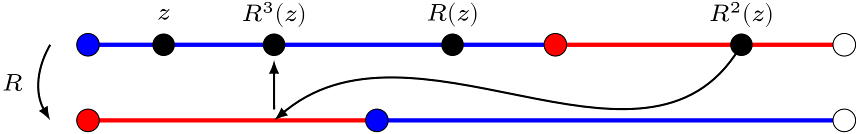


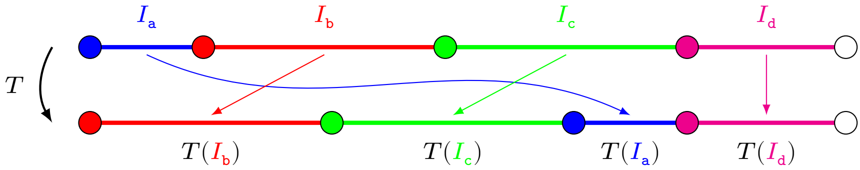


Interval exchanges

Let $(I_a)_{a\in\mathcal{A}}$ be an ordered partition of an interval.

An *interval exchange transformation* (IET) is a map T defined by

$$T(z) = z + \tau_a \quad \text{if} \quad z \in I_a$$



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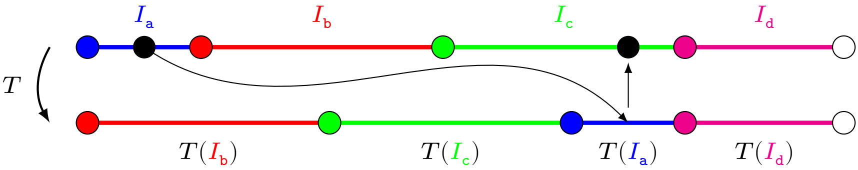

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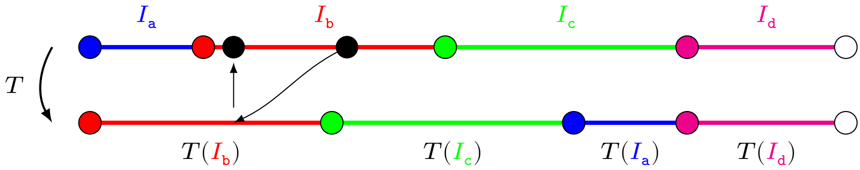

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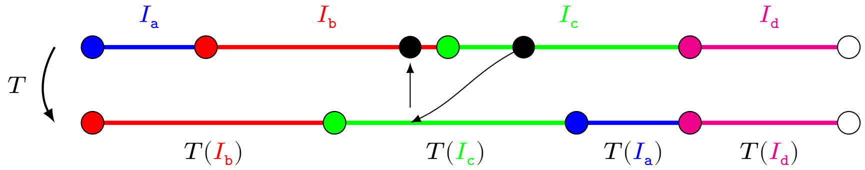

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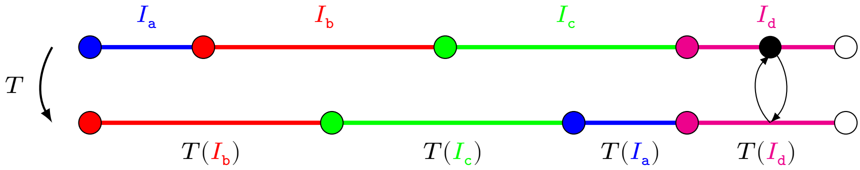

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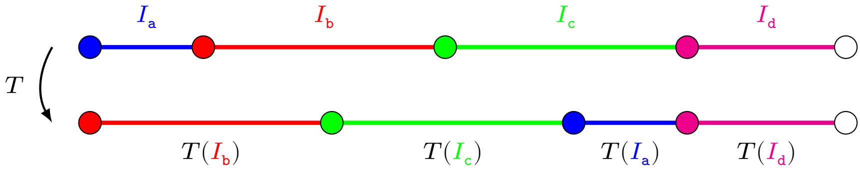


$$\mathcal{A} = \{\mathtt{a} < \mathtt{b} < \mathtt{c} < \mathtt{d}\} \qquad \pi = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} \\ \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{d} \end{pmatrix}$$

Minimality and regularity

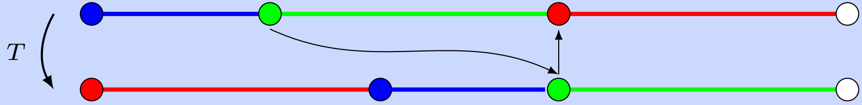
T is *minimal* if every orbit $\mathcal{O}(z) = \{T^n(z) \,|\, n \in \mathbb{Z}\}$ is dense in the interval.

T is *regular* if the orbits of the non-zero formal discontinuities are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)



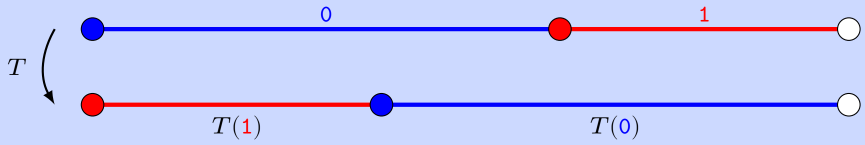
Trajectories

Natural coding

The *trajectory* of z under T is the infinite word $\Omega_T(z) = w_0 w_1 \cdots \in \mathcal{A}^\omega$, defined by

$$w_n = a \qquad \text{if } T^n(z) \in I_a .$$

Example (Fibonacci)



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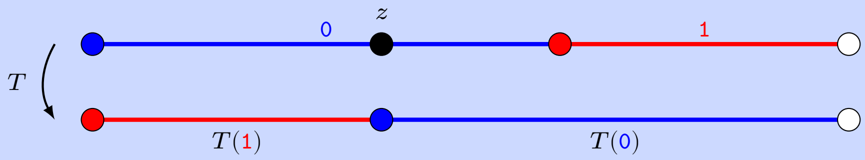

$$\Omega_T(z) = 0$$

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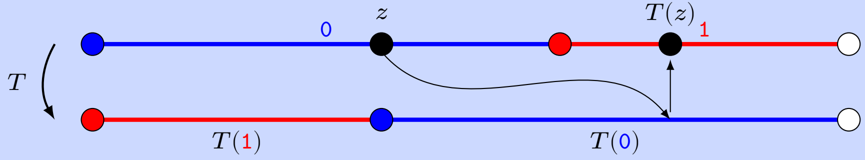

$$\Omega_T(z) = 01$$

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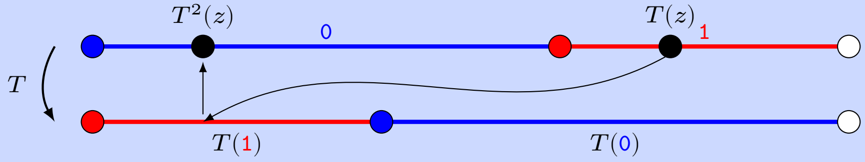

$$\Omega_T(z) = 010$$

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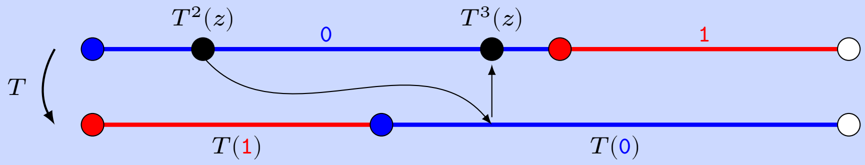

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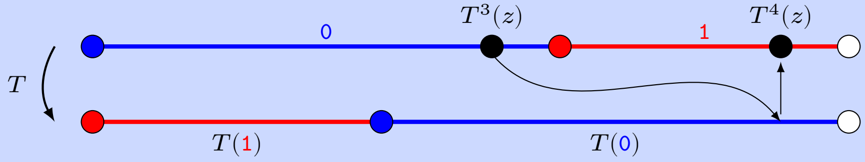

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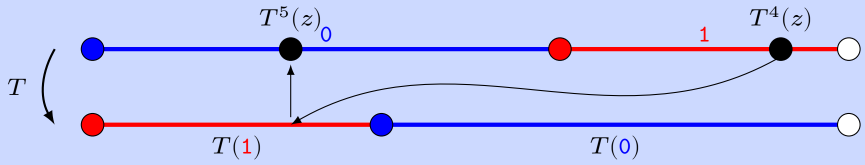


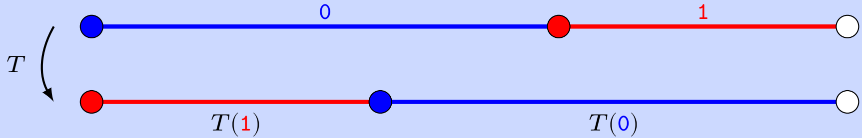
$$\Omega_T(z) = 010010\cdots$$

Interval exchange languages

The set $\mathcal{L}(T) = \bigcup_z \mathcal{L}\left(\Omega_T(z)\right)$ is a (*minimal*, *regular*) *interval exchange language*

Remark. When T is *minimal*, $\mathcal{L}(\Omega_T(z))$ does not depend on the point z.

Example (Fibonacci)



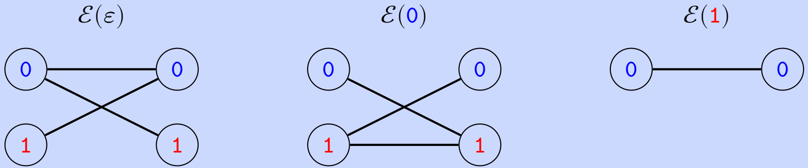
$$\mathcal{L}(T) = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, 0010, \ldots\}$$

Dendric languages

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Regular interval exchange languages are *dendric*.

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, 0, 1, 00, 01, 10, 010, 100, 101, ...\}$)



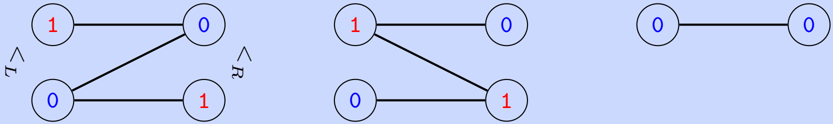
Ordered dendric languages

The graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for $awb,\ cwd \in \mathcal{L}$, one has

$$a <_L c \qquad \Longrightarrow \qquad b \leq_R d.$$

Example (Fibonacci, $1 <_L 0$ and $0 <_R 1$)

$\mathcal{E}(\varepsilon)$ $\qquad$ $\mathcal{E}(0)$ $\qquad$ $\mathcal{E}(1)$



Ordered dendric languages

Example (Tribonacci, $\psi : 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$)

The *Tribonacci* language is **not** ordered dendric.

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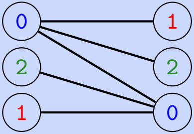

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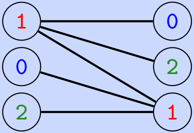
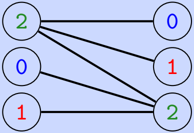


$\mathcal{E}(010)$



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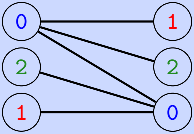

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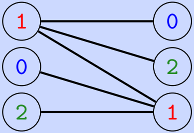

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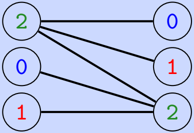

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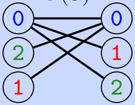

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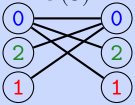

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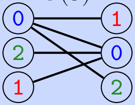

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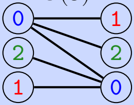

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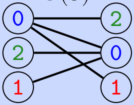

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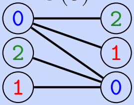
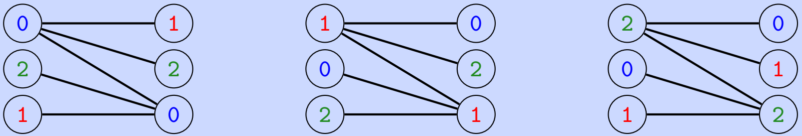


Ordered dendric languages

Example (Tribonacci, $\psi : 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$)

The *Tribonacci* language is **not** ordered dendric.

$\mathcal{E}(\varepsilon)$ $\mathcal{E}(0)$ $\mathcal{E}(010)$



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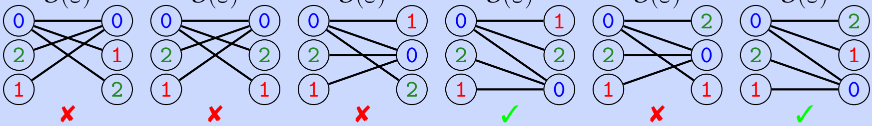

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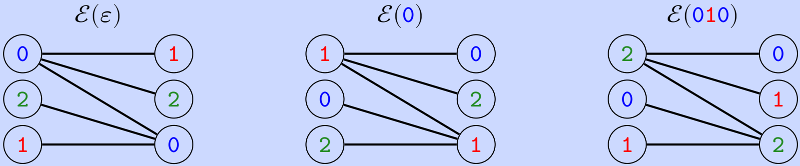

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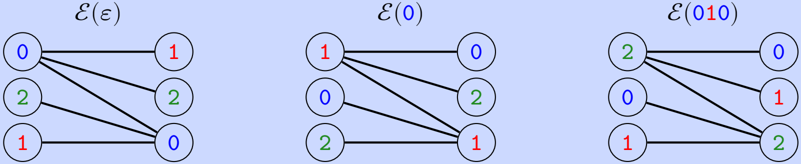

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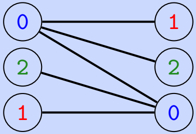

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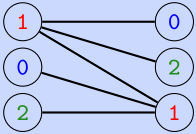

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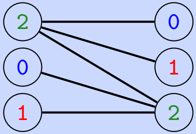

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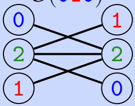

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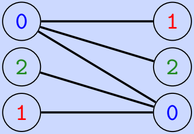

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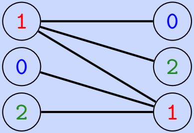

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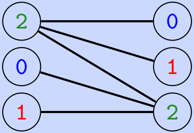

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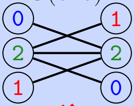
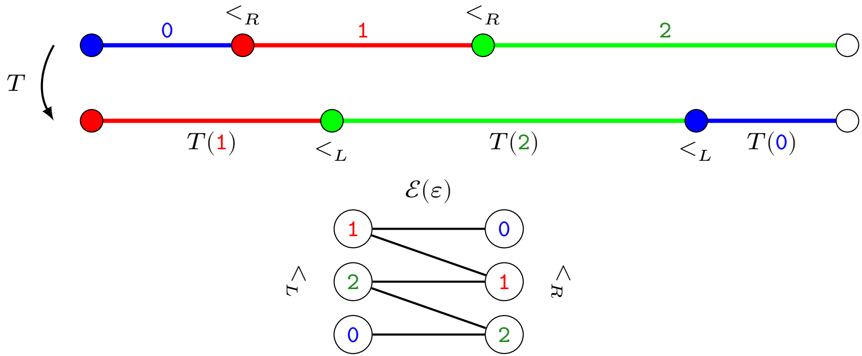


✘

Ordered dendric languages

Theorem [Ferenczi, Zamboni (2008)]

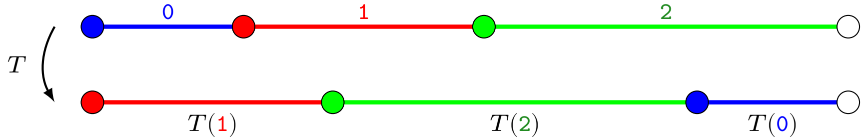
$\mathcal{L}$ is a *regular* interval exchange language iff it is a *recurrent* ordered dendric language.



From letters to words

Given a IET T and a word $w = a_0 a_1 \cdots a_m \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \cdots \cap T^{-m}(I_{a_m})$$



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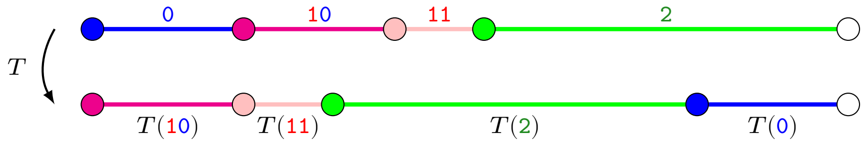

$$I_{10} = I_1 \cap T^{-1}(I_0)$$

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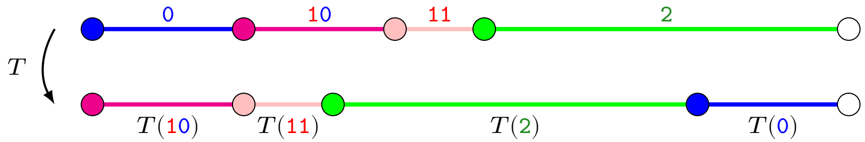

Proposition

- If T is *minimal*, then $w \in \mathcal{L}(T) \Leftrightarrow |I_w| \neq 0$.

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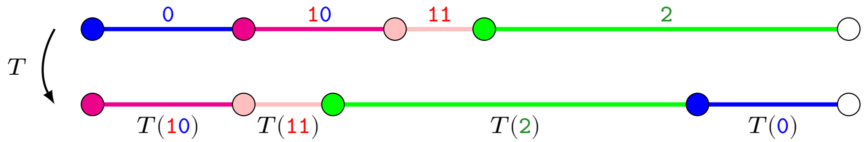
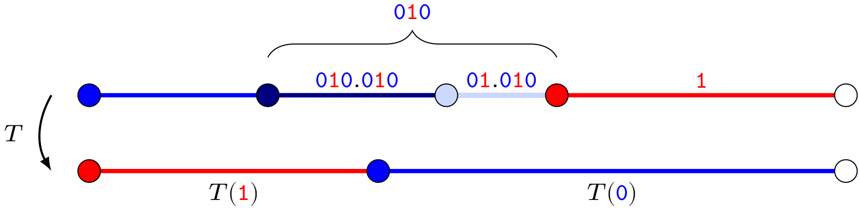


Proposition

- If T is *minimal*, then $w \in \mathcal{L}(T) \Leftrightarrow |I_w| \neq 0$.
- $I_u < I_v$ if and only if $u <_R v$ and u is not a prefix of v.

Return words

$$\mathcal{R}(u) = \{w \in \mathcal{A}^* \mid wu \in (\mathcal{L} \cap u\mathcal{A}^*) \setminus \mathcal{A}^+ u \mathcal{A}^+\}$$



$$\mathcal{R}(010) = \{010,\ 01\}$$

Switching Trick

调包计

qíng ài (情爱) – ài qíng (爱情)

míng tiān (明天) – tiān míng (天明)

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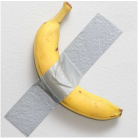
Two words uv and vu are *conjugates*.

$[w] = \{w' = uv \mid w = vu\}$

Burrows-Wheeler Transform



Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.

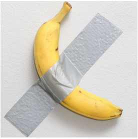
`b a n a n a`

Burrows-Wheeler Transform

Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.

b	a	n	a	n	a					
	a	n	a	n	a	b				
		n	a	n	a	b	a			
			a	n	a	b	a	n		
				n	a	b	a	n	a	
					a	b	a	n	a	n

Burrows-Wheeler Transform



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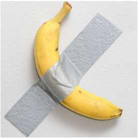
a	b	a	n	a	n
a	n	a	b	a	n
a	n	a	n	a	b
b	a	n	a	n	a
n	a	b	a	n	a
n	a	n	a	b	a

Burrows-Wheeler Transform



Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.

$$
\begin{array}{cccccc}
\mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} \\
\mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{n} \\
\mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{b} \\
\mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} \\
\mathtt{n} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} \\
\mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{b} & \mathtt{a}
\end{array}
$$

$$\mathrm{bwt}_{\mathcal{A}}(\mathrm{banana}) = \mathtt{nnbaaa}$$

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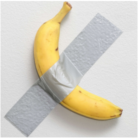

Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.

$$\begin{array}{cccccc}
\mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \color{red}{\mathtt{n}} \\
\mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \color{red}{\mathtt{n}} \\
\mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \color{red}{\mathtt{b}} \\
\mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \color{red}{\mathtt{a}} \\
\mathtt{n} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{n} & \color{red}{\mathtt{a}} \\
\mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{b} & \color{red}{\mathtt{a}}
\end{array}$$

$$\mathrm{bwt}_{\mathcal{A}}(\mathtt{banana}) = \mathtt{nnbaaa}$$

Proposition

w is a conjugate of u^p if and only if $\mathrm{bwt}_{\mathcal{A}}(u) = b_1 \cdots b_n$ and $\mathrm{bwt}_{\mathcal{A}}(w) = b_1^p \cdots b_n^p$.

$$\mathrm{bwt}_{\mathcal{A}}(\mathtt{màn}) = \mathtt{mnà} \qquad \mathrm{bwt}_{\mathcal{A}}(\mathtt{mànmàn}) = \mathtt{mmnnàà}$$

Clustering words

A word $w \in \mathcal{A}^*$ is π-*clustering* for $\mathcal{A}$ if $\mathsf{bwt}_{\mathcal{A}}(w) = a_{\pi(1)}^{k_1} \cdots a_{\pi(d)}^{k_d}$, where $k_i = |w|_{a_{\pi(i)}}$.

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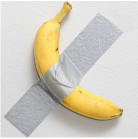
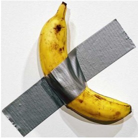
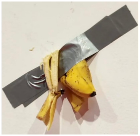
$\mathcal{A} = \{\mathtt{a} < \mathtt{b} < \mathtt{n}\}$	$\mathcal{A}' = \{\mathtt{a} < \mathtt{n} < \mathtt{b}\}$	$\mathcal{A}'' = \{\mathtt{n} < \mathtt{a} < \mathtt{b}\}$
$\mathrm{bwt}_{\mathcal{A}}(\mathtt{banana}) = \mathtt{nnbaaa}$	$\mathrm{bwt}_{\mathcal{A}'}(\mathtt{banana}) = \mathtt{bnnaaa}$	$\mathrm{bwt}_{\mathcal{A}''}(\mathtt{banana}) = \mathtt{aabnna}$
$\pi = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{n} \\ \mathtt{n} & \mathtt{b} & \mathtt{a} \end{pmatrix}$	$\pi' = \begin{pmatrix} \mathtt{a} & \mathtt{n} & \mathtt{b} \\ \mathtt{b} & \mathtt{n} & \mathtt{a} \end{pmatrix}$	not clustering
		

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Over a binary alphabet (perfectly) clustering words are exactly powers of *Christoffel words* (i.e., *finite standard Sturmian words*) and their conjugates.

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And over larger alphabets?

Clustering words and IETs

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Theorem [D., Hughes (2025+)]

Let T be a IET with permutation π. Return words in $\mathcal{L}(T)$ are π-clustering.


谢谢