Interval Exchange Transformations
from Symbolic Dynamics to Combinatorics

Francesco Dolce

Informačne Technológie Aplikácie a Teória

Workshop: Numeration and Substitution Systems

Oravská Lesná, 19. septembra 2020

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from ~~Symbolic Dynamics~~ to ~~Combinatorics~~

Combinatorics *Symbolic Dynamics*

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What to expect from this talk

1. **Combinatorics on Words and Symbolic Dymamics**
   (*A very light introduction to take with your morning coffee*)

2. **Interval Exchange Transformations**
   (*Who are they and what do they want from us?*)

3. **Dendric languages**
   (*Or how to use Greek words to sound more sophisticated*)

4. **Shift spaces**
   (*Entropy, Ergodicity, and other scary words starting in "E"*)

5. **Rauzy induction**
   (*Pardon my French*)

Some words about words

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- $str\check{c}, prst, skrz, krk \in \{a, b, c, \ldots, \check{z}\}^*$

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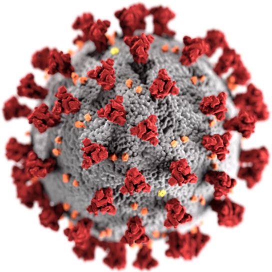
- $\mathtt{ACGATACGGACATTACATATACG} \in \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{T}\}^*$



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- $\mathit{str\check{c}}, \mathit{prst}, \mathit{skrz}, \mathit{krk} \in \{a, b, c, \dots, \check{z}\}^*$
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- $\mathtt{babaabaabaaabaaab} \cdots \in \{\mathtt{a}, \mathtt{b}\}^\omega$

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$$w = pfs \quad \text{with } p, f, s \in \mathcal{A}^* \cup \mathcal{A}^\omega$$

prefix factor suffix

Some words about words

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$$w = pfs \quad \text{with } p, f, s \in \mathcal{A}^* \cup \mathcal{A}^\omega$$

prefix factor suffix

The *language* of w is the set $\mathcal{L}(w) = \{f \mid f \text{ is a factor of } w\}$.

What is a Dymamical System?

$$(X, T)$$

with X a compact metric space and $T : X \to X$ a continuous map.

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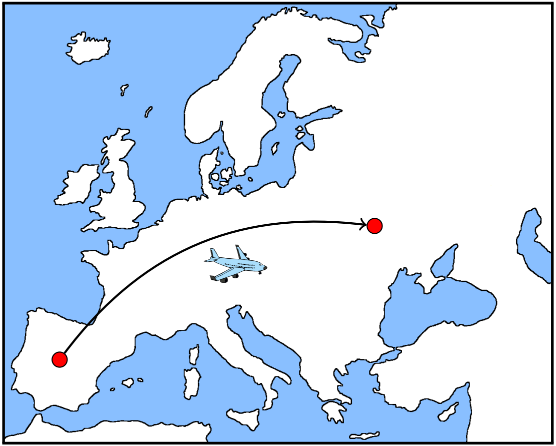

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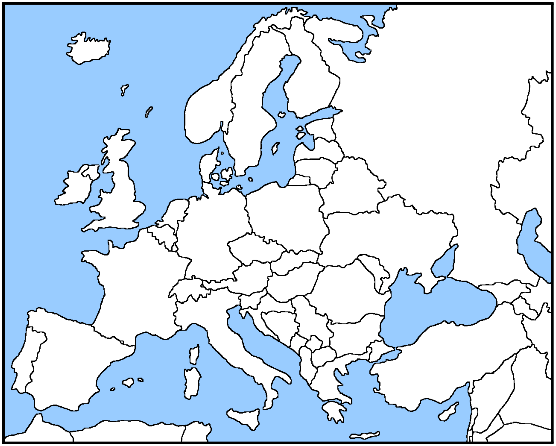

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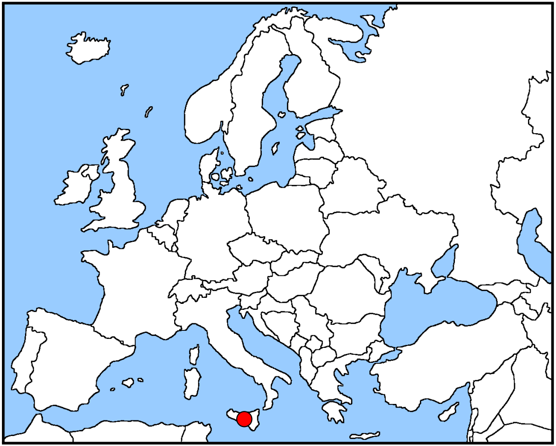

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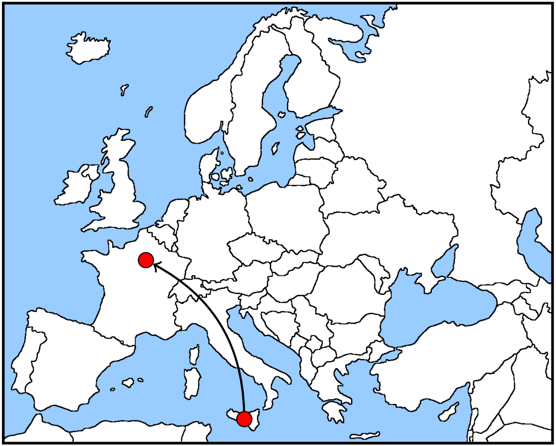

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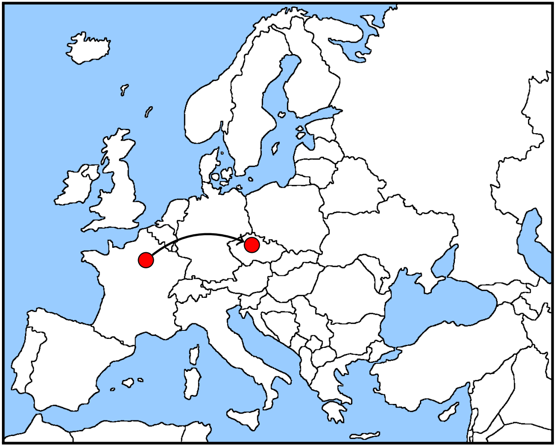

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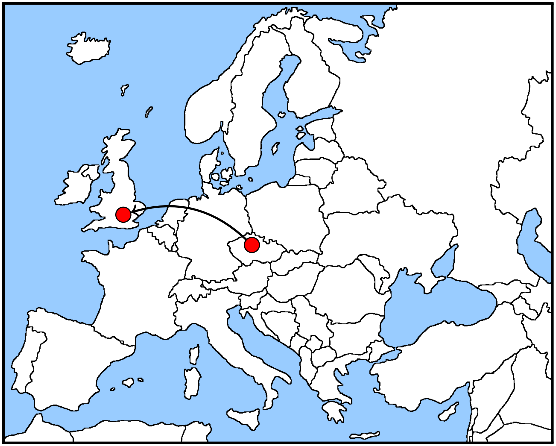

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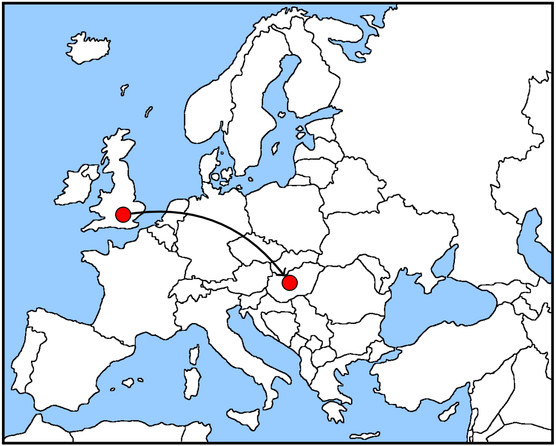

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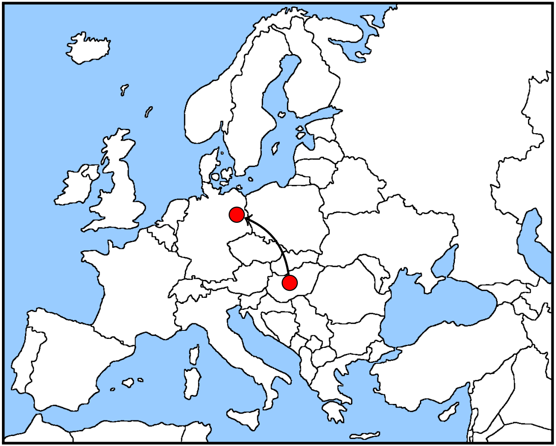

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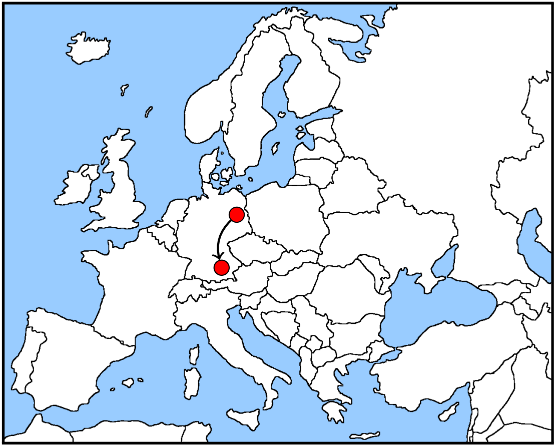

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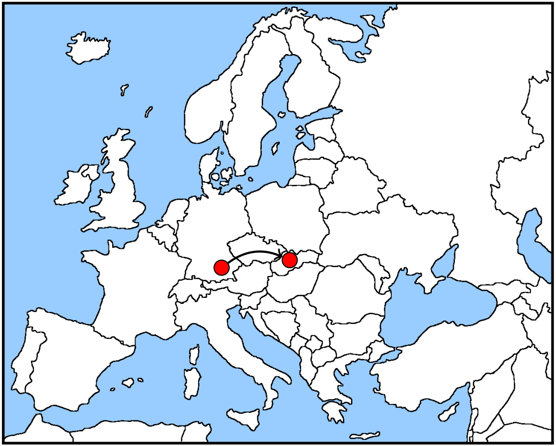
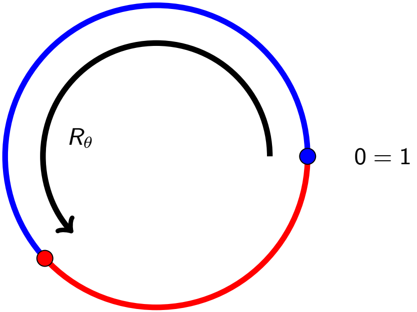


Rotations



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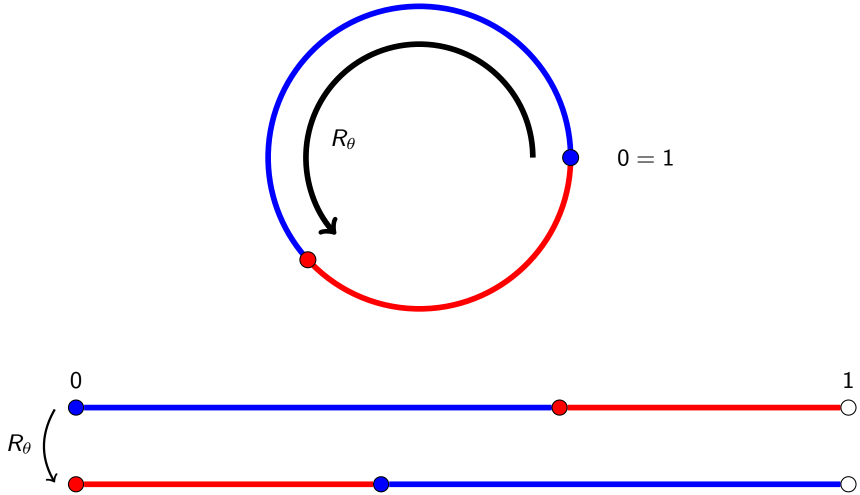

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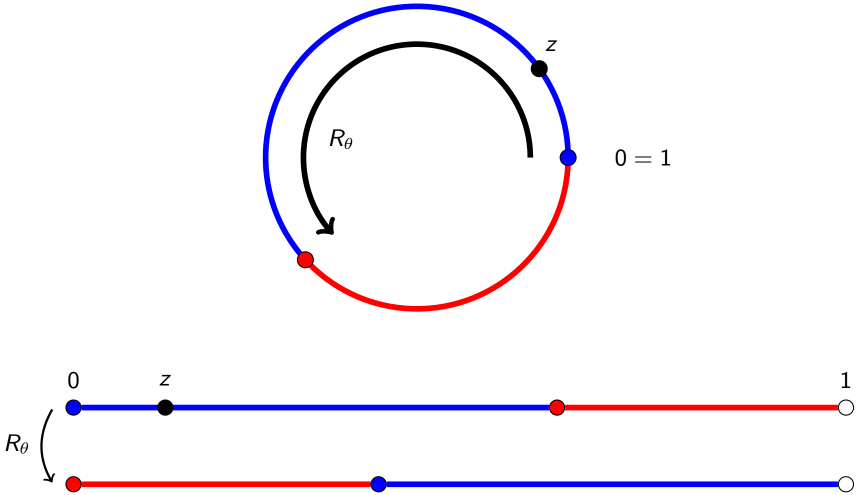

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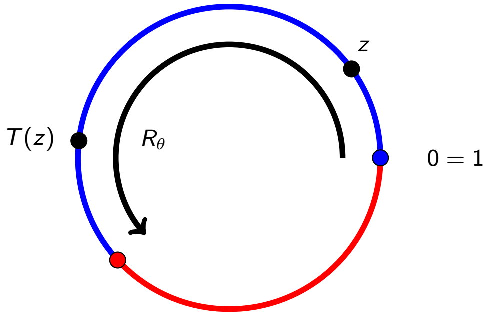

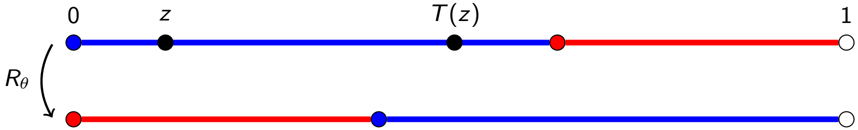

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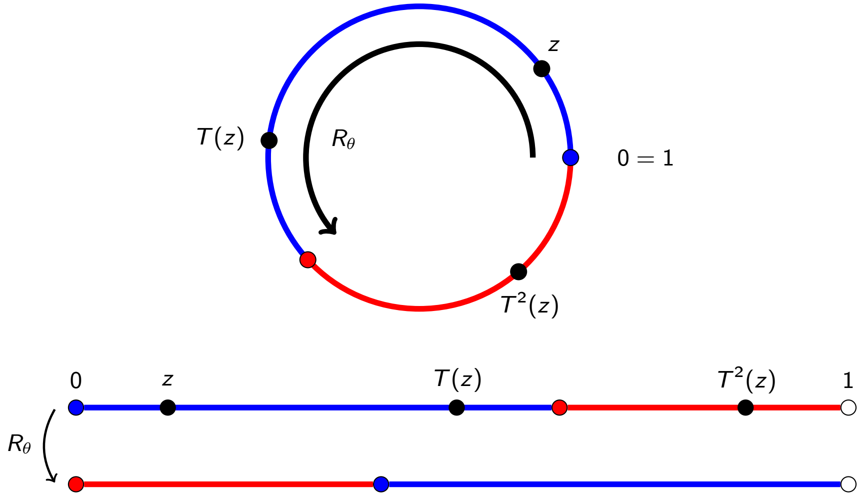

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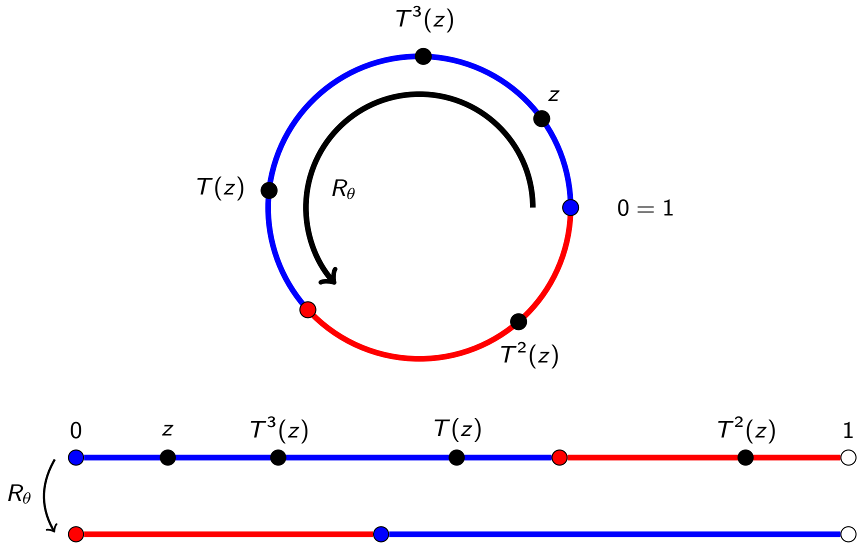
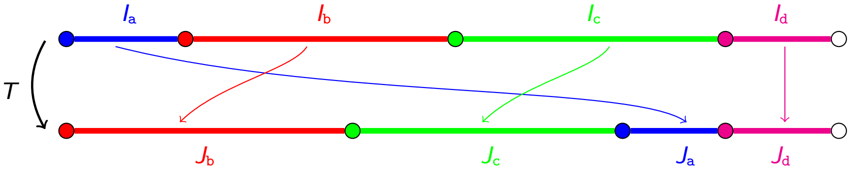


Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$ s.t. $|I_\alpha| = |J_\alpha|$ for every $\alpha \in \mathcal{A}$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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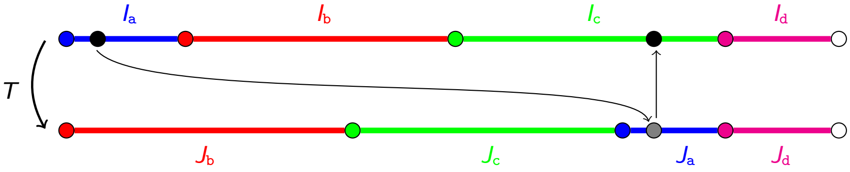

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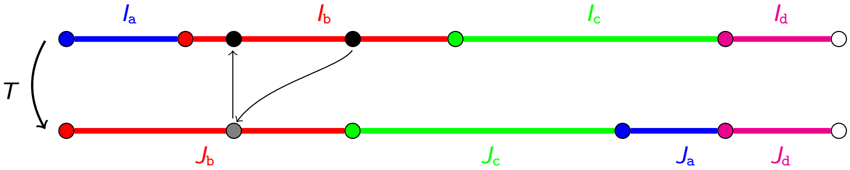

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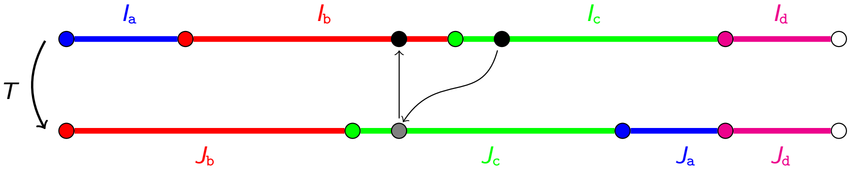

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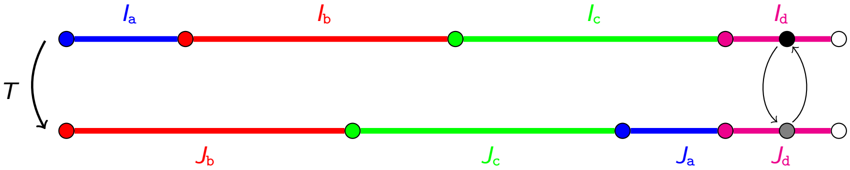

Interval exchanges

T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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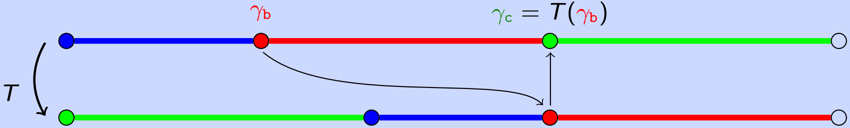
Example (the converse is not true)

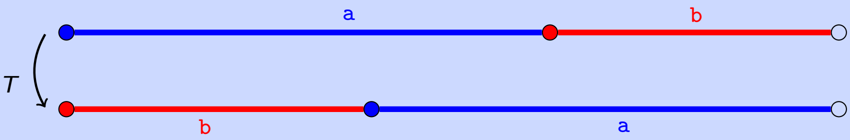


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $\theta = (\sqrt{5}-1)\pi$)



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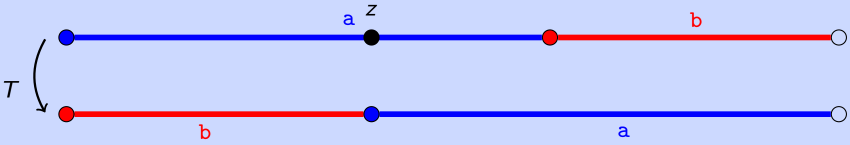

$$\Sigma_T(z) = \text{a}$$

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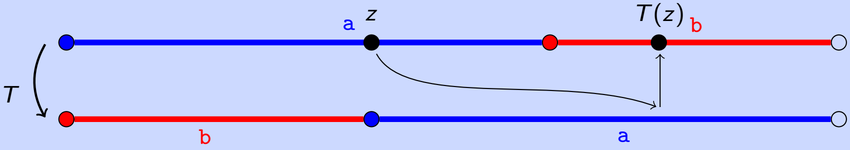

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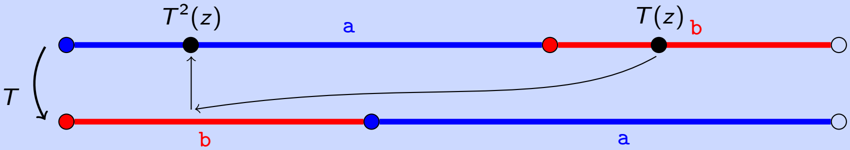

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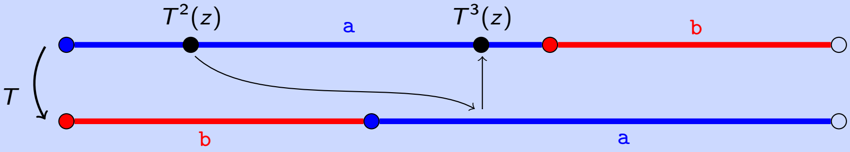

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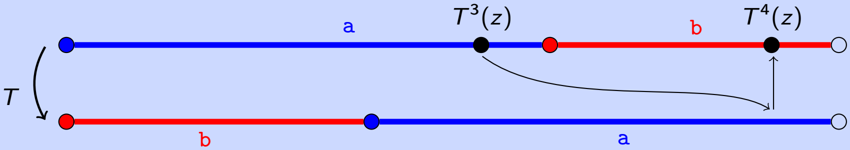

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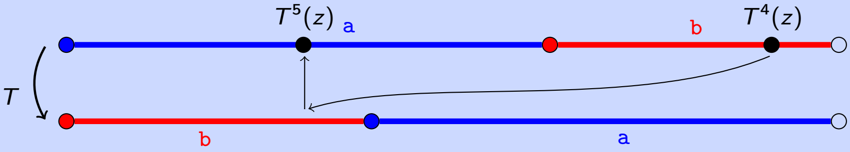

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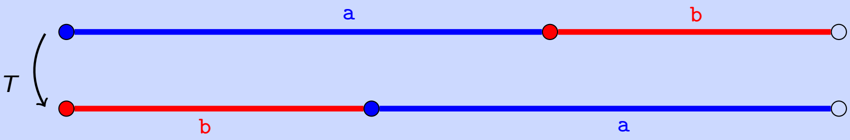
The set $\mathcal{L}(T) = \bigcup_{z\in[0,1[} \mathcal{L}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Example (Fibonacci)



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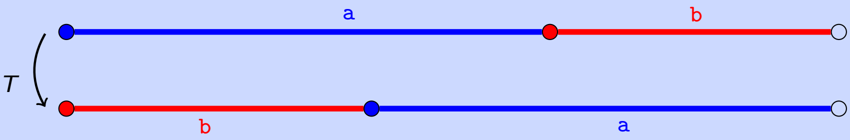

$$\mathcal{L}(T) = \{\varepsilon,\ \mathtt{a}, \mathtt{b},\ \mathtt{aa}, \mathtt{ab}, \mathtt{ba},\ \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \mathtt{aaba},\ \ldots\}$$

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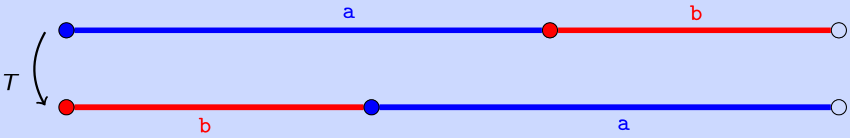

$$\mathcal{L} = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}, \mathtt{c}}_{2}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ba},}_{3}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab},}_{5}, \mathtt{aaba}, \dots\} \qquad p_n = n + 1$$

Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

Example (Fibonacci)

$$x = \mathtt{abaababaabaababaababaabaababaaba} \cdots$$

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Example (Fibonacci)

$$x = \underbrace{\mathtt{abaa}}_{4}\,\mathtt{ba}\,\underbrace{\mathtt{baab}}_{4}\,\underbrace{\mathtt{aaba}}_{4}\,\mathtt{baababaaba}\,\underbrace{\mathtt{abab}}_{4}\,\mathtt{a}\cdots$$

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Uniform recurrence $\implies$ Recurrence.

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Proposition

Uniform recurrence $\Longrightarrow$ Recurrence.

Theorem

A IET T is minimal $\iff$ its language $\mathcal{L}(T)$ is uniformly recurrent.

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$
\begin{aligned}
L(w) &= \{\alpha \in \mathcal{A} \mid \alpha w \in \mathcal{L}\} \\
R(w) &= \{\beta \in \mathcal{A} \mid w\beta \in \mathcal{L}\} \\
B(w) &= \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{A} \mid \alpha w \beta \in \mathcal{L}\}
\end{aligned}
$$

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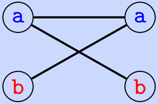
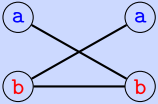
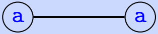


$\mathcal{E}(\text{a})$



$\mathcal{E}(\text{b})$



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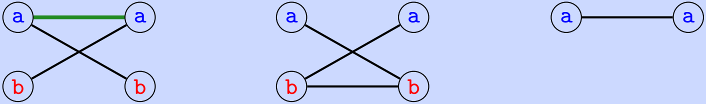

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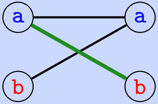
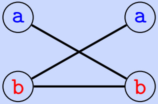


$\mathcal{E}(\mathtt{a})$



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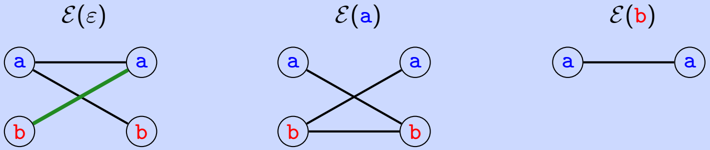

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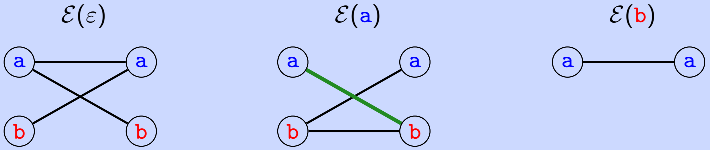

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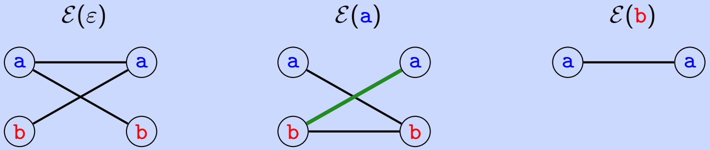

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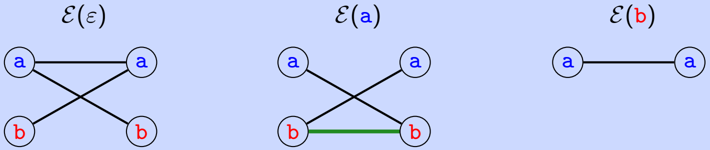

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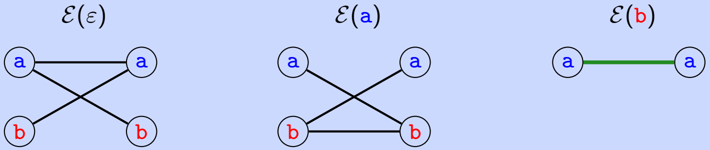

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The *multiplicity* of a word w is the quantity

$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

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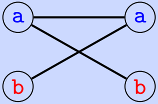
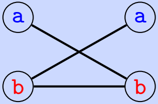
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(\mathrm{a})$



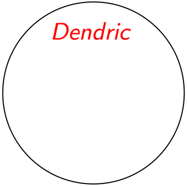
$\mathcal{E}(\mathrm{b})$

$m(\mathrm{a}) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral sets

Definition

A language $\mathcal{L}$ is called *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$.



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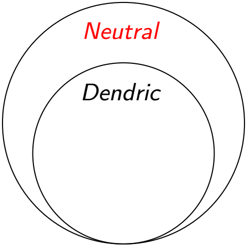

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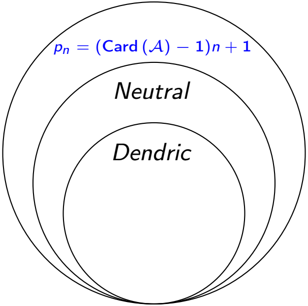

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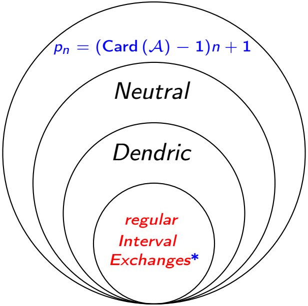

* Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Dendic languages

Arnoux-Rauzy languages

Definition

A language is *Arnoux-Rauzy* if it is closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ and with a unique right special factor for each length.

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Example (Tribonacci)

Factors of the fixed point $\eta^\omega(\mathtt{a})$ of the morphism $\eta : \mathtt{a} \mapsto \mathtt{ab}, \quad \mathtt{b} \mapsto \mathtt{ac}, \quad \mathtt{c} \mapsto \mathtt{a}.$

$$\mathcal{L} = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}, \mathtt{c}}_{3}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{ca}}_{5}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{aca}, \mathtt{baa}, \mathtt{bab}, \mathtt{bac}, \mathtt{cab}}_{7}, \mathtt{aaba}, \ldots\}$$

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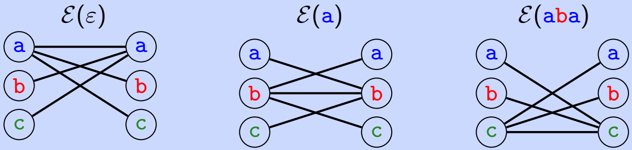

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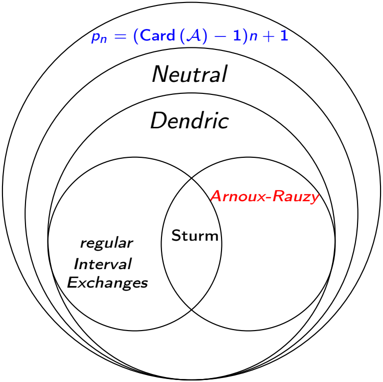
Dendric languages

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Arnoux-Rauzy languages are dendric languages.

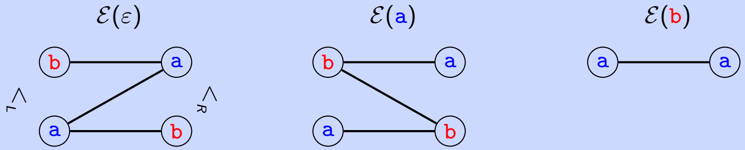


Planar dendric languages

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a language $\mathcal{L}$ and a word $w \in \mathcal{L}$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a,b),(c,d) \in B(w)$, one has

$$a <_L c \quad \Longrightarrow \quad b \leq_R d.$$

Example (Fibonacci, b $<_L$ a and a $<_R$ b)

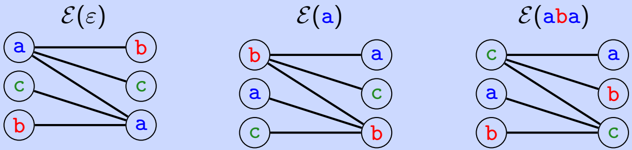


A language $\mathcal{L}$ is *planar dendric* w.r.t. $<_L$ and $<_R$ on $\mathcal{A}$ if for any $w \in \mathcal{L}$ the graph $\mathcal{E}(w)$ is a tree compatible with $<_L$ and $<_R$.

Planar dendric language

Example

The *Tribonacci* language is **not** planar dendric.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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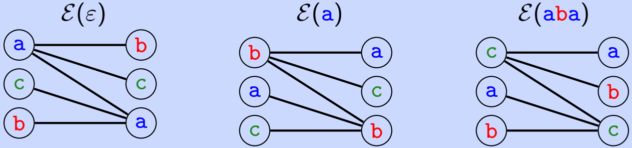
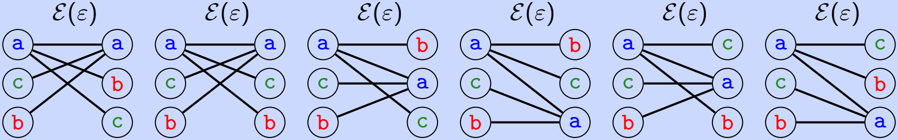


- $\mathtt{a} <_L \mathtt{c} <_L \mathtt{b}$



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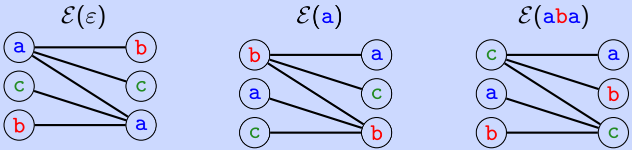
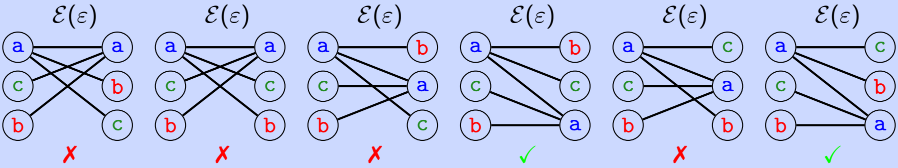


- <u>$\mathtt{a} <_L \mathtt{c} <_L \mathtt{b}$</u> $\Longrightarrow$ $\mathtt{b} <_R \mathtt{c} <_R \mathtt{a}$ or $\mathtt{c} <_R \mathtt{b} <_R \mathtt{a}$



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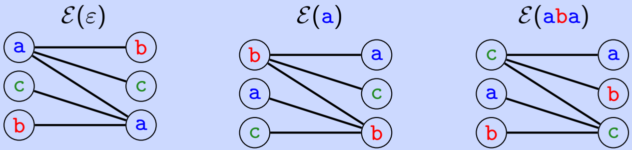

- a $<_L$ c $<_L$ b $\Longrightarrow$ b $<_R$ c $<_R$ a or c $<_R$ b $<_R$ a

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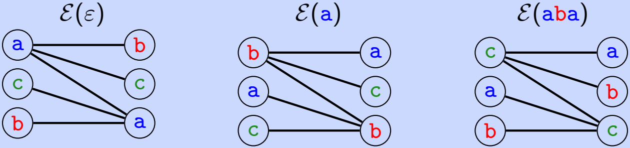

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Planar dendric language

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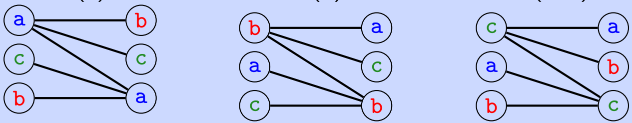
$\mathcal{E}(\varepsilon)$ $\quad$ $\mathcal{E}(\mathtt{a})$ $\quad$ $\mathcal{E}(\mathtt{aba})$



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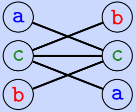
$\mathcal{E}(\mathtt{aba})$



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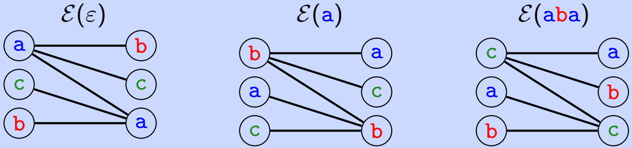

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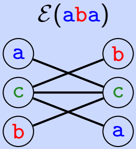
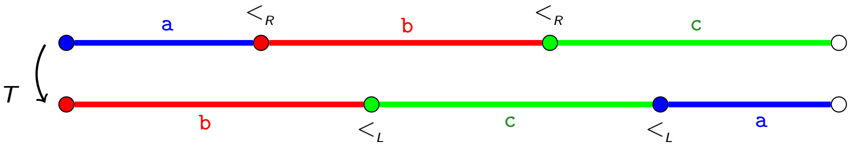
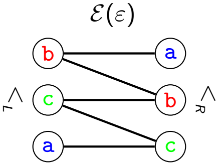


Planar dendric languages

Theorem [Ferenczi, Zamboni (2008)]

A language $\mathcal{L}$ is a regular interval exchange language if and only if it is a recurrent planar dendric language.





Shift spaces

$\mathcal{A}^{\mathbb{Z}} = \{(x_n)_{n\in\mathbb{Z}}\}$ with the natural product topology.
(Equivalently, the compact metric space with distance defined for $(x_n)_n \neq (y_n)_n$ as

$$d\left((x_n)_n, (y_n)_n\right) = \frac{1}{\min\{|i| \geq 0 \mid x_i \neq y_i\}}.$$

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The *shift transformation* is the function

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

Example (Fibonacci)

$$\mathbf{x} = \cdots \mathtt{ab.abaababaabaababaababaabaab} \cdots$$

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$$\begin{array}{rcl} \mathbf{x} & = & \cdots ab.abaababaabaababaababaabaab \cdots \\ \sigma(\mathbf{x}) & = & \cdots ba.baababaabaababaababaabaaba \cdots \end{array}$$

Shift spaces

$\mathcal{A}^{\mathbb{Z}} = \{(x_n)_{n\in\mathbb{Z}}\}$ with the natural product topology.
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The pair (X, σ), with X a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *shift space*.

Example (Fibonacci, but on two sides)

The *Fibonacci shift space* is the set $X = \overline{\mathcal{O}(\mathbf{x})} = \overline{\{\sigma^k(\mathbf{x}) \mid k \in \mathbb{Z}\}} \subset \mathcal{A}^{\mathbb{Z}}$, with

$$\mathbf{x} = \cdots \mathtt{ab.abaababaabaababaababaabaab} \cdots$$

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A shift space (X, σ) is a *dendric shift* (a *IET shift*) if its language is dendric (a IET set).

Entropy of dendric shift space

The *entropy* of a shift (X, σ) having language $\mathcal{L}(X) \subset \mathcal{A}^*$ is defined as

$$h(X) = \lim_{n\to\infty} \frac{\log(p_n)}{n} \ (\geq 0)$$

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Proposition

All dendric shift spaces have entropy zero.

Ergodicity of dendric shift spaces

A probability measure μ on (X, σ) is said to be *invariant* if $\mu\left(\sigma^{-1}(U)\right) = \mu(U)$ for every Borel subset U of X.

A shift space having only one invariant probability measure is said to be *uniquely ergodic*.

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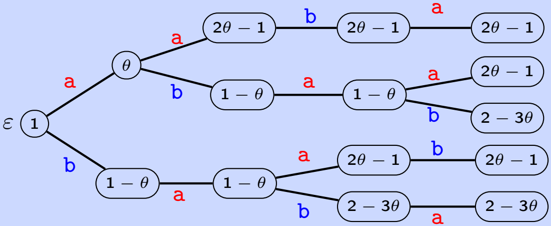
Theorem [P. Arnoux, G. Rauzy (1991)]

Shift spaces associated to Arnoux-Rauzy sets are uniquely ergodic.

Example (Fibonacci, $\theta = (\sqrt{5} - 1)/2$)



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Given an interval exchange transformation T and a word $w = a_0 a_1 \cdots a_{m-1} \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \ldots \cap T^{-m+1}(I_{a_{m-1}})$$

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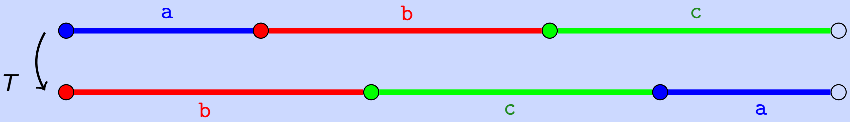

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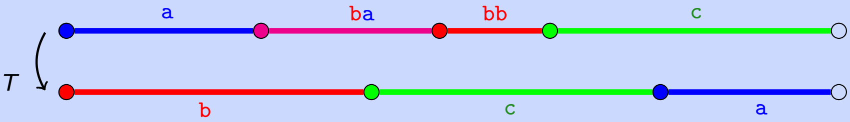

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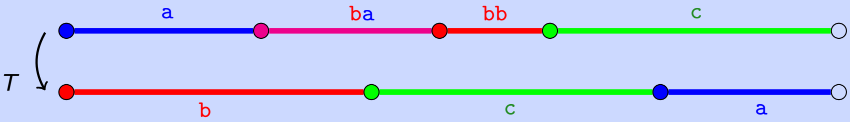

The map μ defined by $\mu([w]) = |I_w|$ is an invariant probability measure.

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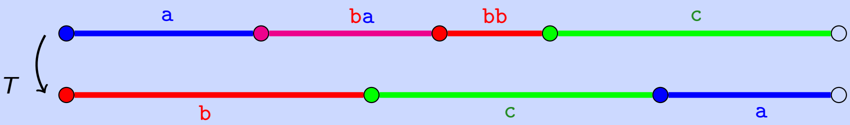

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QUESTION : Is it the only one?

Ergodicity of dendric shift spaces

Conjecture [Keane (1975)]

Every regular IE is uniquely ergodic.

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Corollary

Dendric shift spaces are **not** in general uniquely ergodic (even when minimal).

Ergodicity of dendric shift spaces

Theorem [Boshernitzan (1984)]

A minimal symbolic system such that $\limsup_{n\to\infty}\left(\frac{p_n}{n}\right) < 3$ is uniquely ergodic.

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Minimal dendric shift spaces over an alphabet of size ≤ 3 are uniquely ergodic.


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Minimal dendric shift spaces over an alphabet of size ≤ 3 are uniquely ergodic.

Theorem [Damron, Fickenscher (2019)]

A minimal dendric shift space has at most $\frac{\sup_n(p_{n+1} - p_n) + 1}{2}$ ergodic measures.

Induced transformations

Let T be a minimal interval exchange transformation and $I \subset [0, 1[$.
The *transformation induced* by T on I is the transformation $S : I \to I$ defined by

$$S(z) = T^n(z) \qquad \text{with } n = \min\{k > 0 \mid T^k(z) \in I\}.$$

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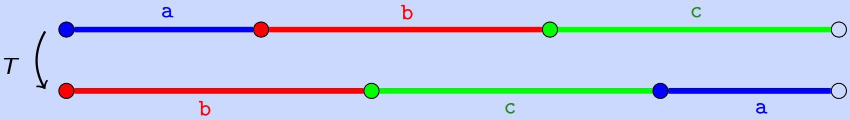

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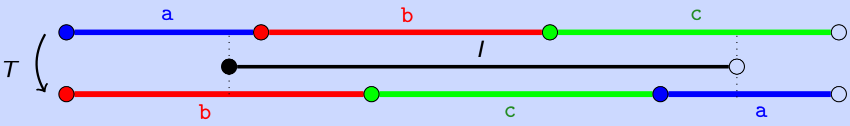

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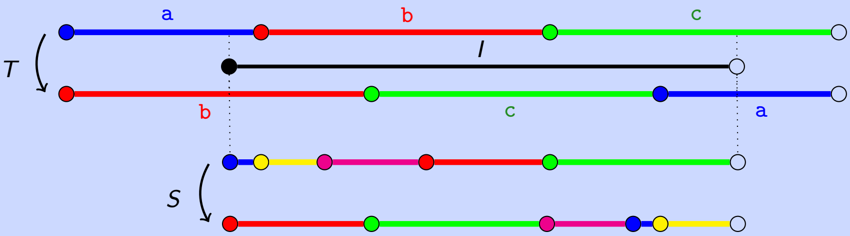

Right-admissible semi-intervals

A semi-interval $[0, r[$, with $0 < r < 1$, is *right-admissible* for T if there exists a $k \in \mathbb{Z}$, s.t. $r = T^k(\gamma_\alpha)$ for some $\alpha \in \mathcal{A}$ and:

(*i*) if $k > 0$, then $r < T^h(\gamma_\alpha)$ for all $0 < h < k$,

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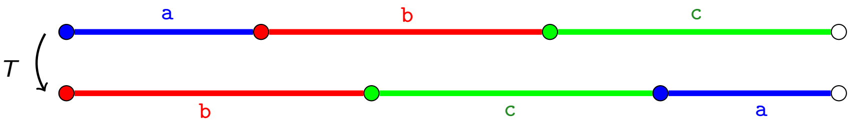

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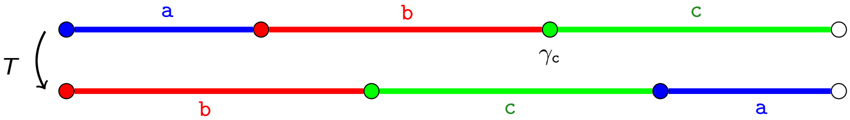

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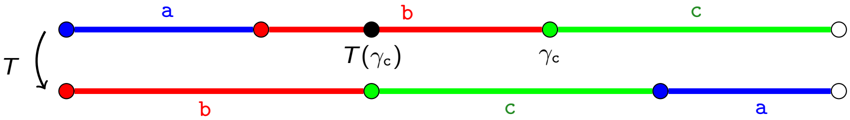

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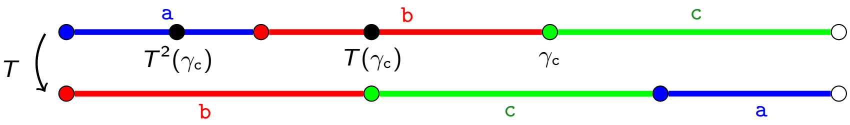

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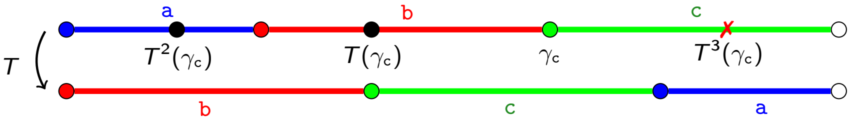

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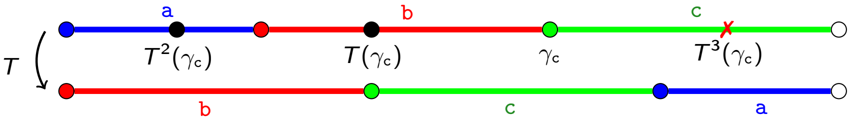


Theorem [Rauzy (1979)]

Let T a regular IET and I a right-admissible interval for T.
Then, the induced transformation by T on I is a regular IET (on the same alphabet).

Right Rauzy induction

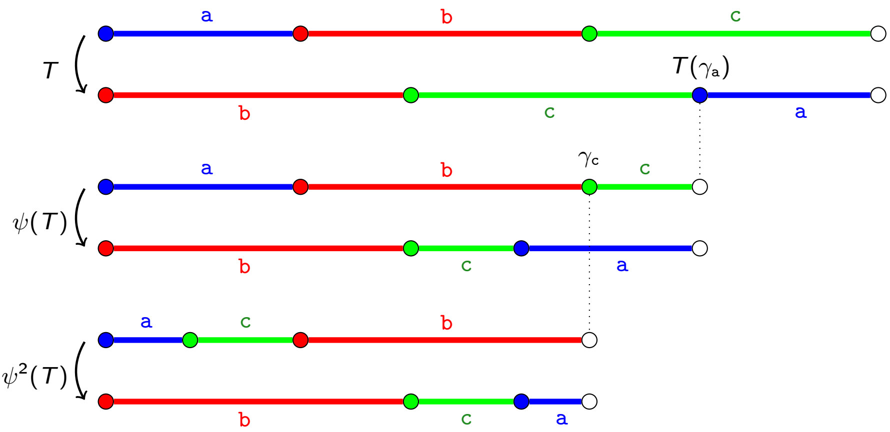
Given a regular IET T, set $Z(T) = \left[0, \max_{\alpha \in A}\{\gamma_\alpha,\ T(\gamma_\alpha)\}\right[$.
We denote by $\psi(T)$ the transformation induced by T on $Z(T)$.

Theorem [Rauzy (1979)]

Let T a regular IET.
A semi-interval I is right-admissible for $T \iff I = Z\left(\psi^n(T)\right)$ for some $n > 0$.
In this case, the transformation induced by T on I is $\psi^{n+1}(T)$.

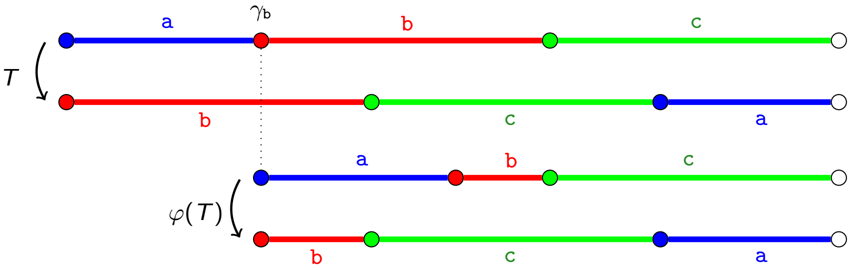
The map $T \to \psi(T)$ is called *right Rauzy induction*.

Right Rauzy induction



Left Rauzy induction

The notions of *left admissible semi-interval* and the *left Rauzy induction* $T :\to \varphi(T)$ are defined simmetrically.



Two-sided Rauzy induction

A semi-interval $[\ell, r[$, with $0 \leq \ell < r \leq 1$ is *admissible* for a regular IET T if $\ell, r \in \mathrm{Div}(I, T) \cup \{1\}$, with

$$\mathrm{Div}(I, T) = \bigcup_{\alpha \in \mathcal{A}} \left\{ T^k(\gamma_\alpha) \mid -\rho^-(\gamma_\alpha) \leq k < \rho^+(\gamma_\alpha) \right\}$$

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The interval I_w is admissible for every $w \in \mathcal{L}(T)$.

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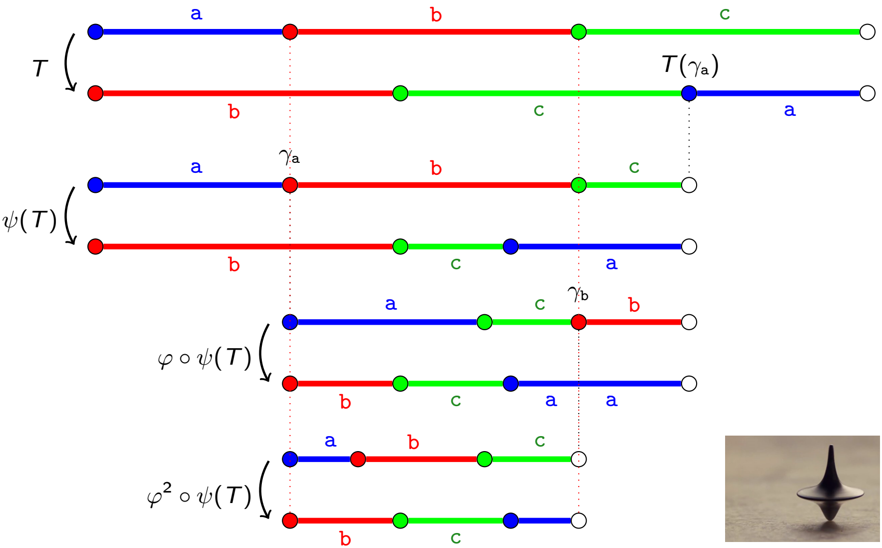

Rauzy induction and Euclidean algorithm

Let $\mathbb{R}_+^2 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq 0,\ \lambda_2 \geq 0\}$, and the map $\mathcal{E} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ given by

$$\mathcal{E}(\lambda_1, \lambda_2) = \begin{cases} (\lambda_1 - \lambda_2,\ \lambda_2) & \text{if } \lambda_1 \geq \lambda_2 \\ (\lambda_1,\ \lambda_2 - \lambda_1) & \text{if } \lambda_1 < \lambda_2 \end{cases}$$

Rauzy induction and Euclidean algorithm

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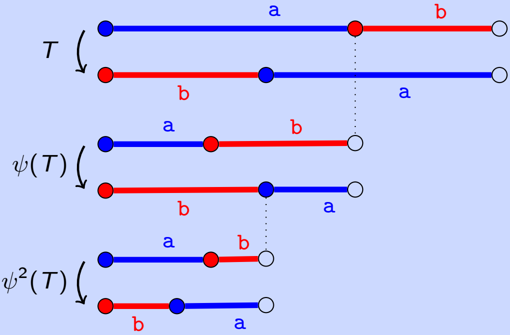
Example



$$\frac{|I_a|}{|I_b|} = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

Ďakujem za pozornosť