

Rauzy Induction

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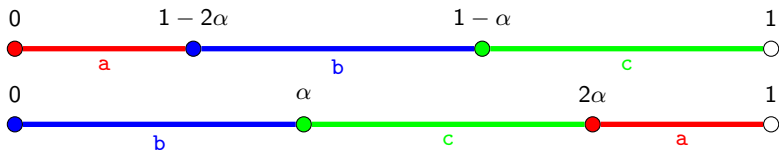
Seminář o IET slovech
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Interval Exchange Transformations

A d -IET is determined by :

- an alphabet $\mathcal{A}$,
- a length vector $\lambda = (\lambda_a)_{a \in \mathcal{A}}$,
- a pair of permutations (π_t, π_b) .

The permutation $\pi = \pi_b \circ \pi_t^{-1}$ is called the *monodromy invariant* of T .



$$\lambda = (1 - 2\alpha, \alpha, \alpha), \quad \pi = (1, 3, 2)$$

π_t	a	b	c
	1	2	3
π_b	b	c	a

Minimality and regularity

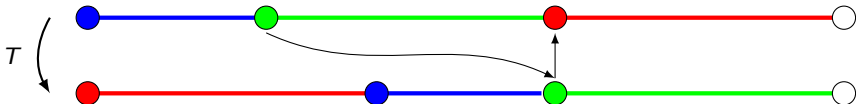
T is *minimal* if for any $z \in [0, 1)$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1)$.

T is *regular* if the orbits of the non-zero formal discontinuities are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

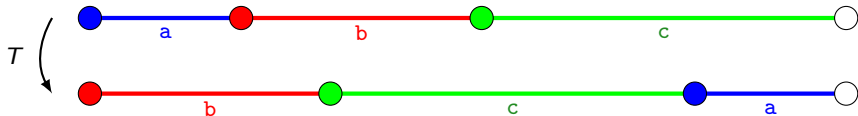
The converse is not true.



From letters to words

Given a IET T and a word $w = w_0 w_1 \cdots w_m \in \mathcal{A}^*$, let

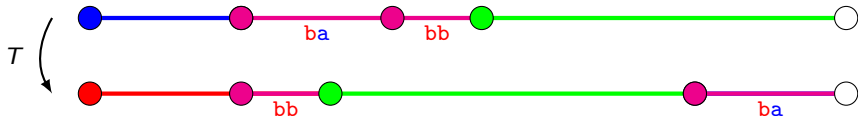
$$I_w = I_{w_0} \cap T^{-1}(I_{w_1}) \cap \cdots \cap T^{-m}(I_{w_m}) \quad (\text{by convention } I_\varepsilon = [0, 1))$$



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$$I_{ba} = I_b \cap T^{-1}(I_a) \quad T^2(I_{ba}) = T^2(I_b) \cap T(I_a)$$

Thus $\Omega_T(z)$ starts with w for every $z \in I_w$.

Induced transformations

Let T be an IET and $J \subset [0, 1)$.

The *transformation induced* by T (*first return map* of T) on J is $T' : J \rightarrow J$ defined by

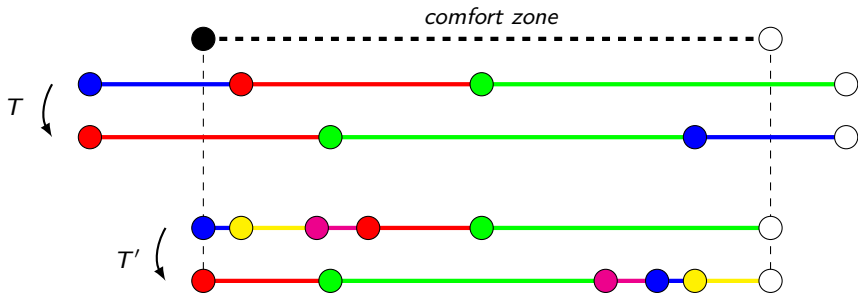
$$T'(z) = T^{\nu(z)}(z) \quad \text{with } \nu(z) = \min\{n > 0 \mid T^n(z) \in J\}$$

Induced transformations

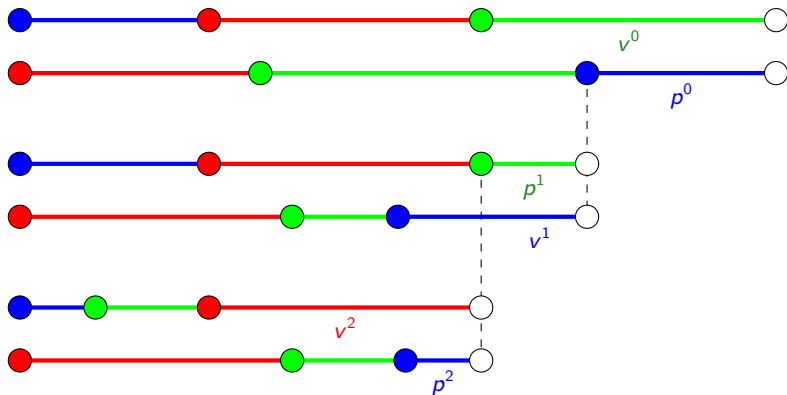
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(Right) Rauzy induction



$$\lambda_{v^n}^{n+1} = \lambda_{v^n}^n - \lambda_{p^n}^n$$

(Right) Rauzy induction and regularity

Theorem [Rauzy (1979)]

If T is *regular*, the (right) Rauzy induced transformation is *regular* on the same alphabet.

Theorem

T is regular **iff** the (right) Rauzy induction starting with (I, T) can be indefinitely iterated.

In this case, $(v^n)_n$ and $(p^n)_n$ take every value infinitely many times; and $|I^n| \rightarrow 0$.

(Right) Rauzy classes

$k = 2, 3$

$$t \curvearrowleft \begin{pmatrix} a & b \\ b & a \end{pmatrix} \curvearrowright b$$

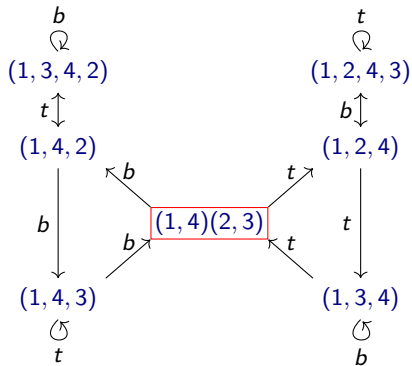
(1, 2)

$$t \curvearrowleft \begin{pmatrix} a & c & b \\ c & b & a \end{pmatrix} \xleftrightarrow{b} \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \xleftrightarrow{t} \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \curvearrowright b$$

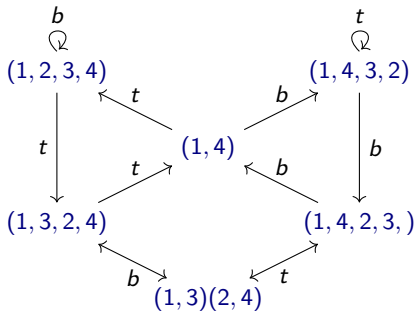
(1, 3, 2) (1, 3) (1, 2, 3)

(Right) Rauzy classes

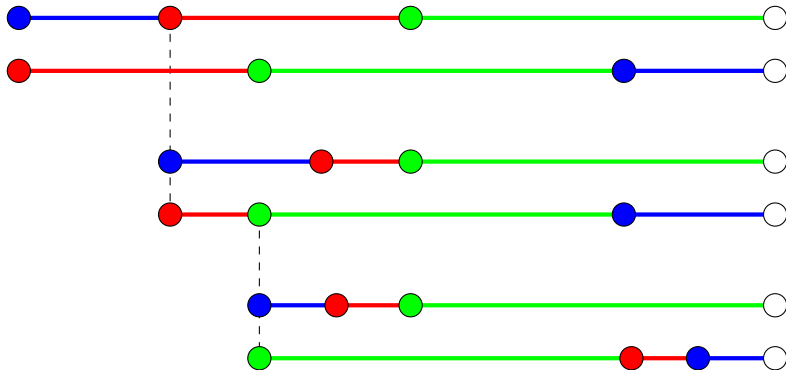
$$k = 4$$



hyperelliptic Rauzy class

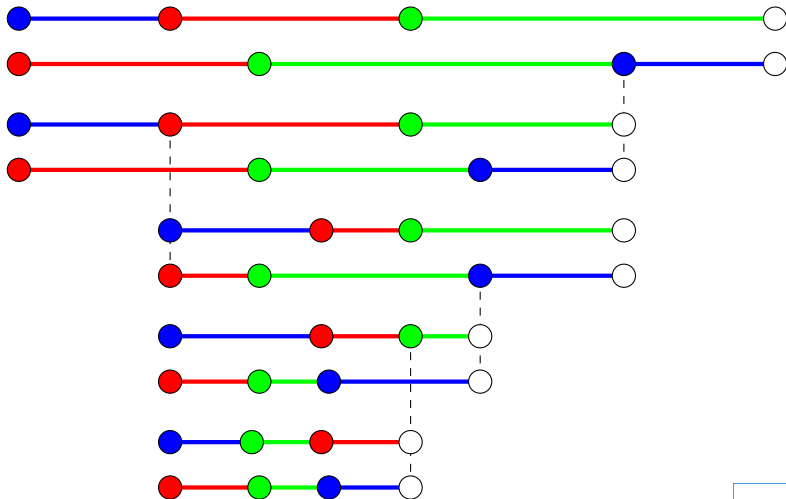


Left Rauzy induction



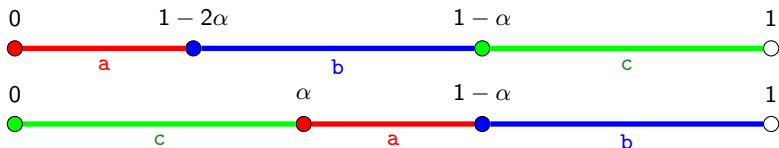
Branching Rauzy induction

Going two-sided



Branching induction and Extended Rauzy classes

Branching Rauzy induction may be defined indefinitely even for some T not regular !



An extended Rauzy class may contain several Rauzy classes.

$$\pi = \begin{pmatrix} \textcolor{red}{a} & \textcolor{blue}{b} & \textcolor{green}{c} & \textcolor{violet}{d} & \textcolor{yellow}{e} \\ \textcolor{yellow}{e} & \textcolor{green}{c} & \textcolor{blue}{b} & \textcolor{violet}{d} & \textcolor{red}{a} \end{pmatrix} \quad \text{and} \quad \pi' = \begin{pmatrix} \textcolor{red}{a} & \textcolor{blue}{b} & \textcolor{green}{c} & \textcolor{violet}{d} & \textcolor{yellow}{e} \\ \textcolor{yellow}{e} & \textcolor{violet}{d} & \textcolor{blue}{b} & \textcolor{green}{c} & \textcolor{red}{a} \end{pmatrix}$$

belong to distinct (right) Rauzy classes, but to the same extended class.

More on Rauzy classes

Every (right) Rauzy class is strongly connected.

$$N_k = k! - \sum_{i=1}^{k-1} (k-i)! N_i.$$

k	1	2	3	4	5	6	7
N_k	1	1	3	13	71	461	29096

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Theorem [Fickenscher (2017)]

Every (right, extended) Rauzy class contains an involution.

Corollary

Every (right, extended) Rauzy class is closed under taking inverses.



Rauzy induction on I_w

Theorem [D., Perrin (2017)]

Let T be a regular IET and $w \in \mathcal{L}(T)$.

The regular IET T' induced on I_w can be obtained by two-sided Rauzy induction.

Moreover, each step is associated with an automorphism of $F_{\mathcal{A}}$ of the form

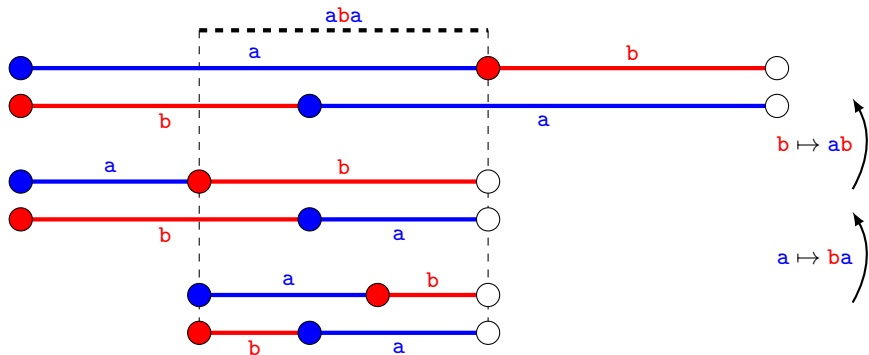
$$\alpha_{a,b} = \begin{cases} a \mapsto ab \\ c \mapsto c \end{cases} \quad \text{or} \quad \tilde{\alpha}_{a,b} = \begin{cases} a \mapsto ba \\ c \mapsto c \end{cases}$$

and for every $z \in I_w$

$$\Omega_T(z) = \theta(\Omega_{T'}(z)).$$

(θ is obtained from the morphisms above, but backwards!)

Rauzy induction on I_w



$$\theta : \begin{cases} a \mapsto ba \mapsto aba \\ b \mapsto b \mapsto ab \end{cases}$$

$$\mathcal{R}(aba) = \{aba, ab\}$$

Closure by derivation

Theorem

Let T be regular k -IET. For every $w \in \mathcal{L}(T)$, $D_w(T)$ is a regular k -IET language.

Inducing T on l_c we have :

$$\mathcal{R}(c) = \{\underbrace{cba}_1, \underbrace{cbba}_2, \underbrace{c}_3\} \quad \text{and} \quad \mathcal{R}'(c) = \{bac, bbac, c\}$$

