Lyndon words

Francesco Dolce

Konference Combinatorics on Words

Janov nad Nisou, 12. září 2020

Just to fix the notation

In case you were distracted during Vašek's talk

Definition

- $\mathcal{A}$ (finite) *alphabet*
- $a \in \mathcal{A}$ *letter*
- $w \in \mathcal{A}^*$ *(finite) word*
- $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$
- $w^n = \underbrace{ww \cdots w}_{n}$, with the convention $w^0 = \varepsilon$
- $w = pfs$ with $p, f, s \in \mathcal{A}^*$, p *prefix*, f *factor*, s *suffix* (*proper* • if in $\mathcal{A}^+$)

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- $w = pfs$ with $p, f, s \in \mathcal{A}^*$, p *prefix*, f *factor*, s *suffix* (*proper* • if in $\mathcal{A}^+$)
- $|w|$ *length* of w, $|w|_a = \#a$ in w, $|w| = \sum_{a \in \mathcal{A}} |w|_a$
- $w[0, k)$ is the prefix of length k of w $w = w[0, |w|)$

Borders and Primitiveness

Definition

- If $w = us = pu$, then u is a called a *border* of w, and w is said to be *bordered*. If no such p exists, w is said to be *unbordered*.

Example

- acbaabaac, kombinatorik, ababa.

Borders and Primitiveness

Definition

- If $w = us = pu$, then u is a called a *border* of w, and w is said to be *bordered*. If no such p exists, w is said to be *unbordered*.
- A word w is said to be *primitive* if $\left[w = u^n \implies n = 1 \text{ and } w = u\right]$

Example

- acbaabaac, kombinatorik, ababa.
- a, b ab, babaa, ~~aaa~~, ~~abab~~

Congiugate words

Definition

Two words $w, w' \in \mathcal{A}^+$ are *conjugate*, denoted $w \equiv w'$, if there exist $p, s \in \mathcal{A}^+$ s.t. $w = ps$ and $w' = sp$.

The *class of conjugacy* of w is $[w] = \{w' \mid w' \equiv w\}$

Example

- $\mathtt{aba} \equiv \mathtt{aab}$, $\mathtt{abab} \equiv \mathtt{baba}$.
- $[\mathtt{aba}] = \{\mathtt{aab}, \mathtt{aba}, \mathtt{baa}\}$, $[\mathtt{abab}] = \{\mathtt{abab}, \mathtt{baba}\}$.

Oredered alphabets

Definition

Let us consider a total order $<$ on $\mathcal{A}$.
This order can be extended to $\mathcal{A}^*$, and it is called *lexicographical order*, by setting

$$u < v \iff \begin{array}{ll} v = us & s \in \mathcal{A}^* \\ \text{or} & \\ u = pas,\ v = pbt & p, s, t \in \mathcal{A}^*,\ a, b \in \mathcal{A},\ a < b \end{array}$$

Example

If $\mathcal{A} = \{a, b, c\}$ and $a < b < c$, then

$$\mathtt{a} < \mathtt{aab} < \mathtt{ab} < \mathtt{aba} < \mathtt{b} < \mathtt{bac} < \mathtt{bb}.$$

Lyndon words

Definition [R. Lyndon (1954), А. И. Ширшов (1953)]

$w \in \mathcal{A}^+$ is a *Lyndon word* (or *правильное слово*) if for all $p, s \in \mathcal{A}^+$ s.t. $w = ps$ one has one of the three following equivalent conditions:

1. $w < sp$,
2. $w < s$,
3. $p < s$.

Example

a, b, ab, aab, ababb, ~~abab~~, ~~ba~~ .

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Proposition

A word w is Lyndon word iff w is primitive and smaller than all its conjugates.

Lyndon factorization

Theorem [Lyndon (1954)]

Each word $w \in \mathcal{A}^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i Lyndon word for every i and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n$.

Example

- aacab
- bc.bc.a
- b.abb.ab.a
- ab.a.a
- b.aaac.a

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Theorem [Duval (1980)]

The Lyndon factorization can be computed in linear time.

Proof. [*idea of*]

(∃)
- ▷ Each word has a trivial (maybe increasing) factorization in Lyndon words: $w = a_0 a_1 \cdots a_{|w|-1}$, with $a_i \in \mathcal{A}$.
- ▷ If u, v are Lyndon words and $u < v$, then uv is also Lyndon.

(!) If $w = \ell_1 \ell_2 \cdots \ell_n$ is the Lyndon factorization of w, then

- ▷ ℓ_1 is the longest prefix which is Lyndon.

Duval's algorithm

The algorithm uses three positions $i < j \leqslant k$ such that:

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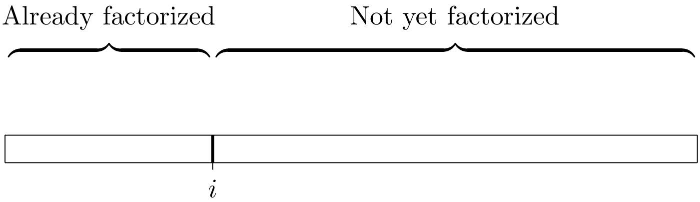
- $w[0, i)$ has already been factorized in $u_1^{m_1} \cdots u_s^{m_s}$



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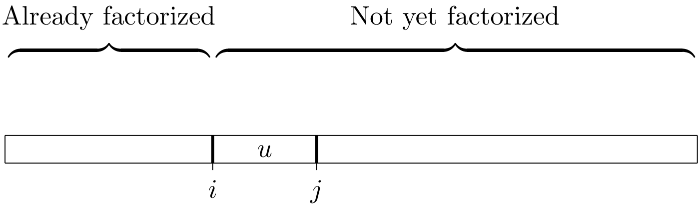

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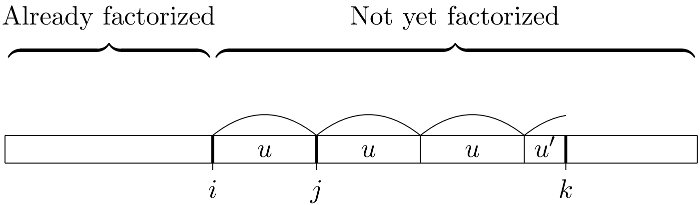
- $u = w[i, j)$ is a Lyndon word: candidate for the next factor
- $w[i, k) = u^m u'$ where $m \geqslant 1$ and u' is a prefix of u



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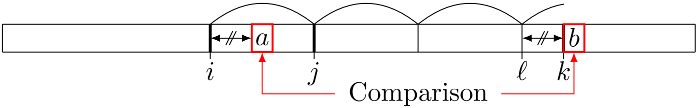
Duval's algorithm (continued)



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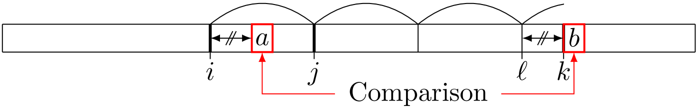
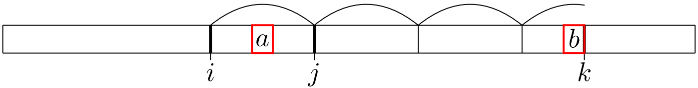


Case $a = b$: $k \leftarrow k + 1$



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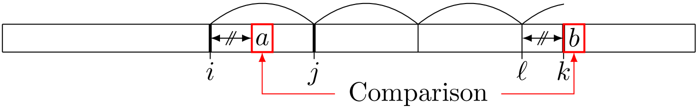

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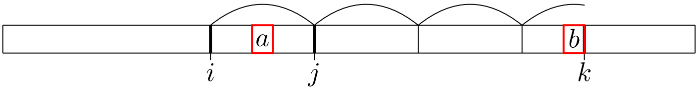

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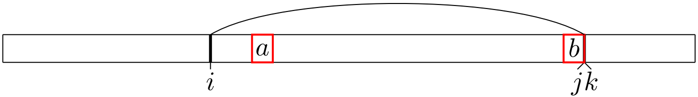

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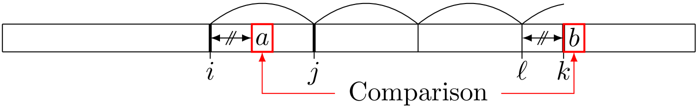
Duval's algorithm (continued)



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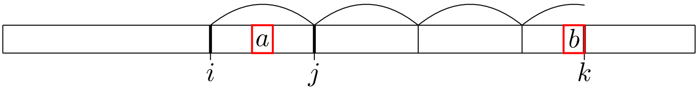

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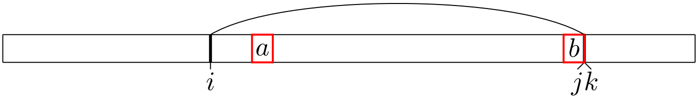
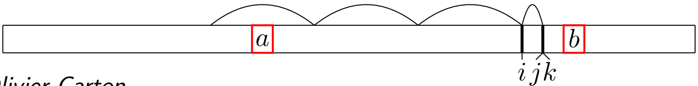


Case $a > b$: $i \leftarrow \ell$; $j \leftarrow \ell + 1$; $k \leftarrow \ell + 1$



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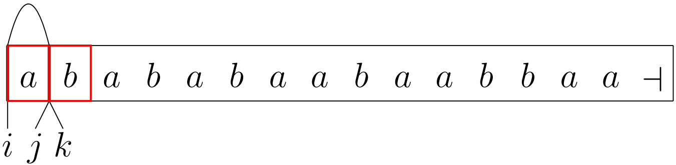
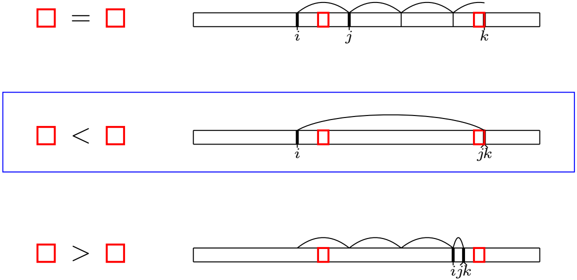
Example





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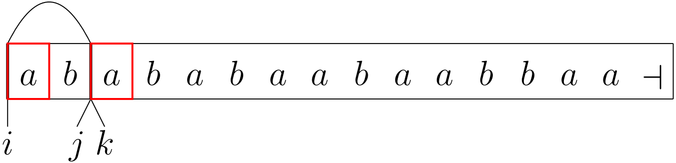

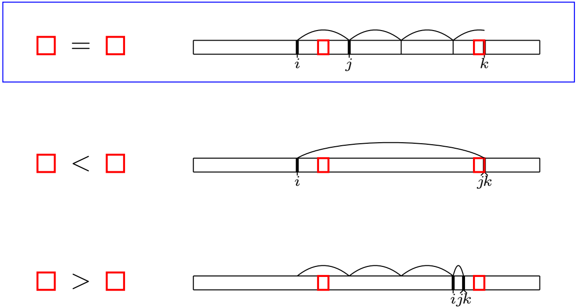

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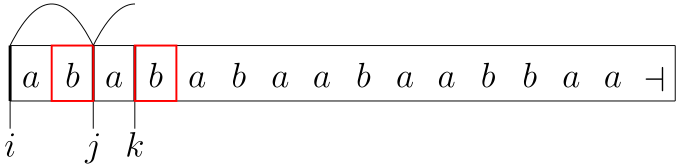

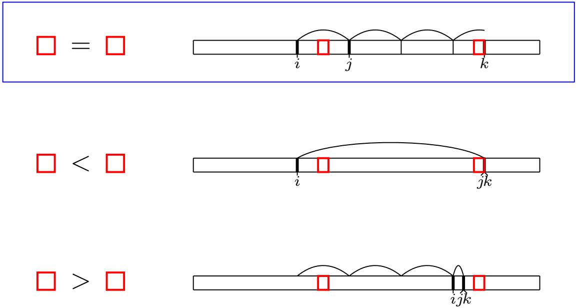

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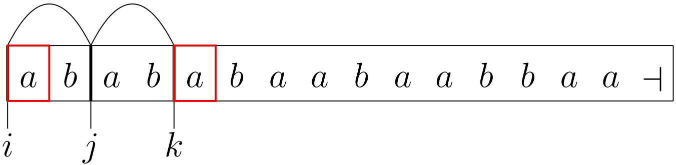

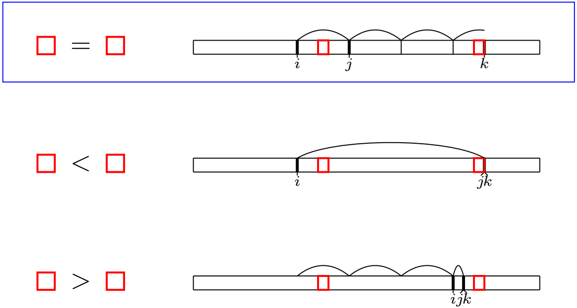

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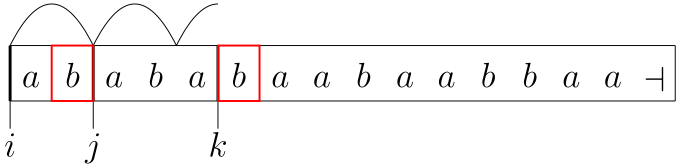

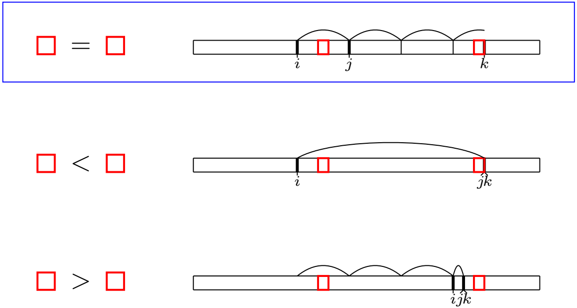

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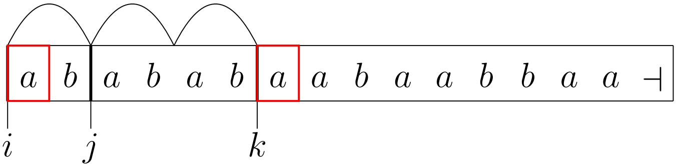

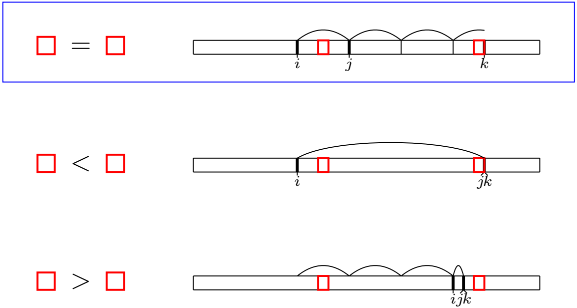

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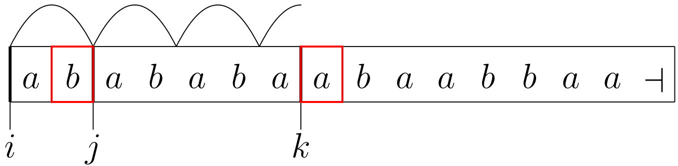

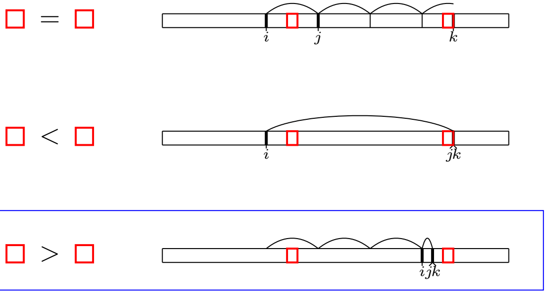

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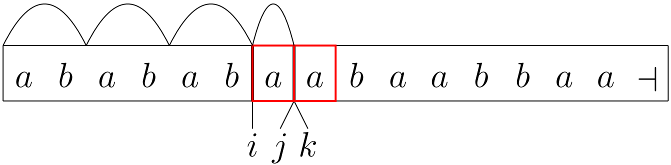

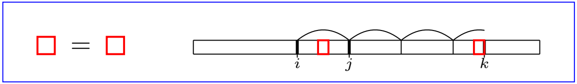

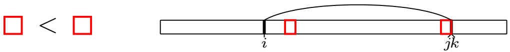

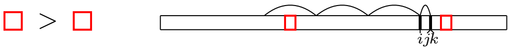

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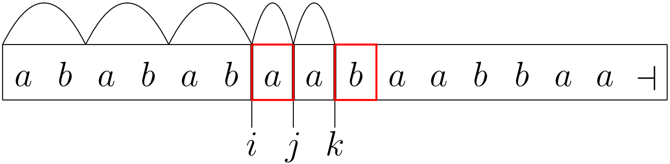

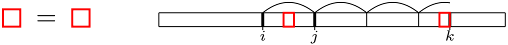

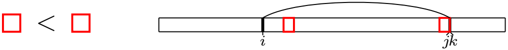

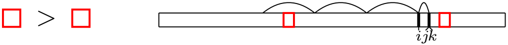

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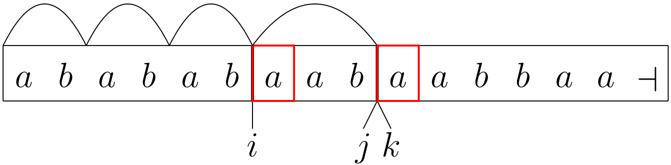

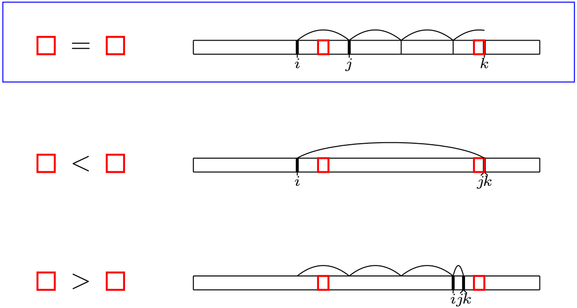

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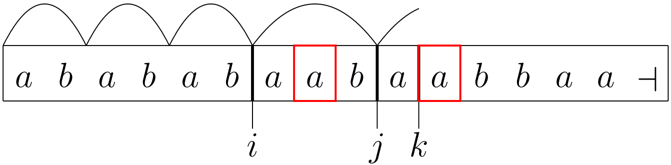

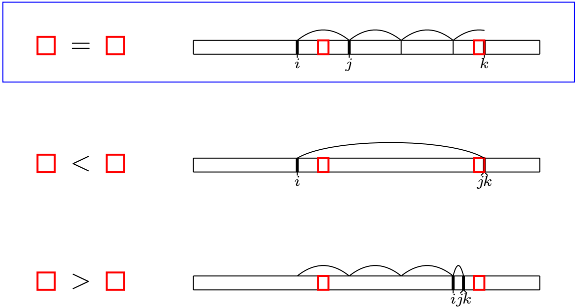

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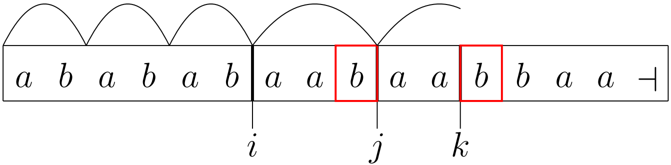

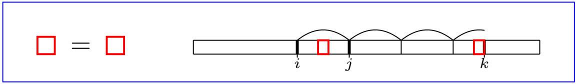

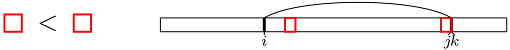

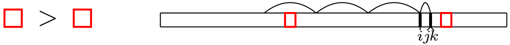

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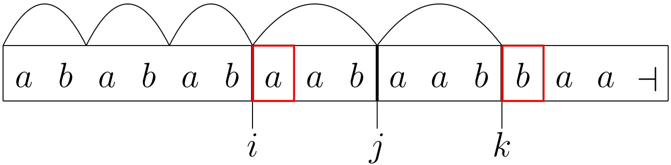

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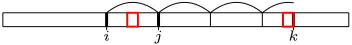

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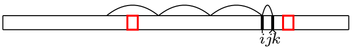

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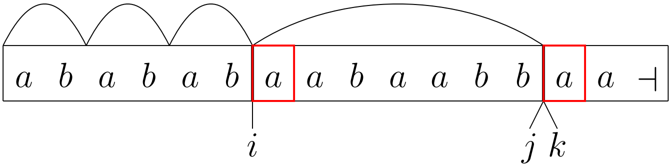

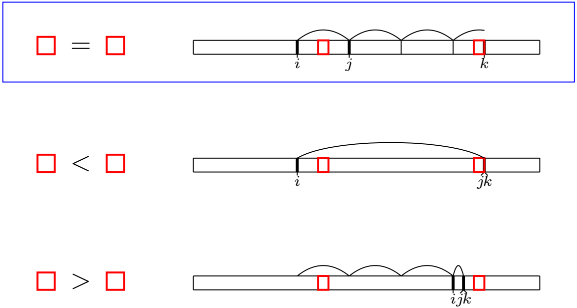

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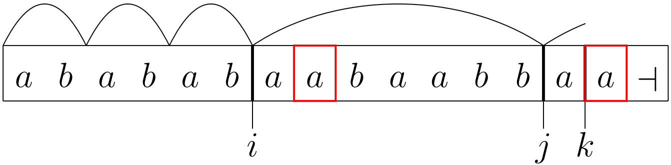
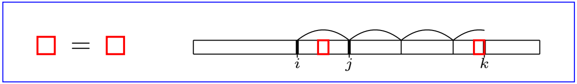
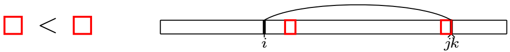
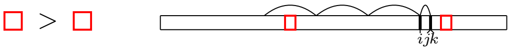


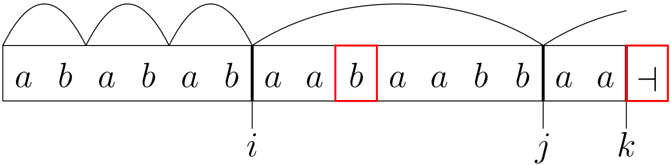


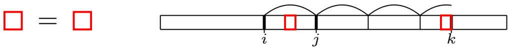


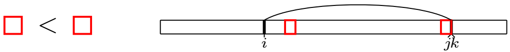


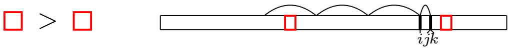
Credits: *Olivier Carton*

Example









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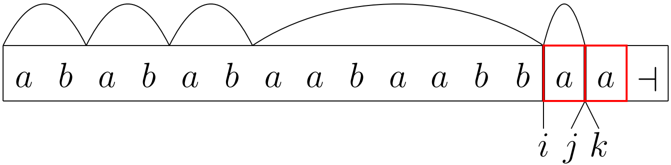

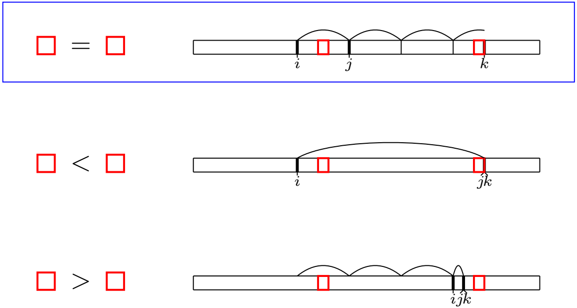

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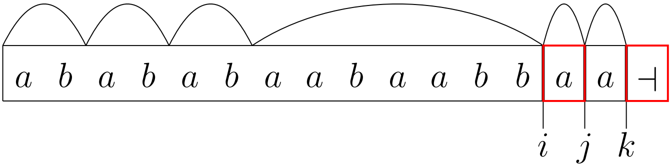

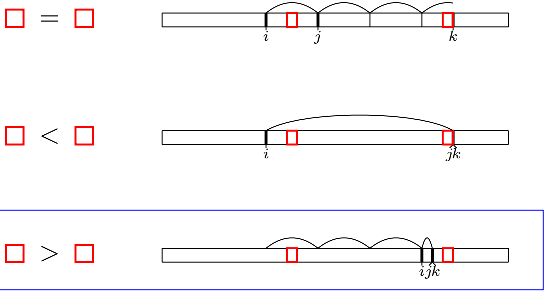

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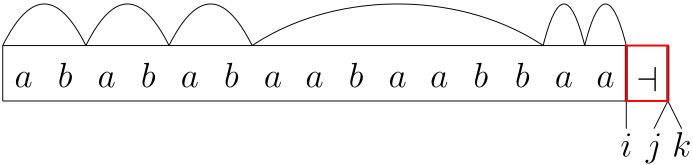

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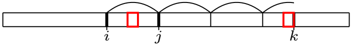

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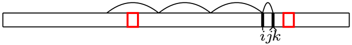

Credits: *Olivier Carton*

To infinity and beyond

Infinite words

Definition

- $\mathbf{w} = a_0 a_1 a_2 \cdots$ is an *infinite word*
- $\mathcal{A}^\omega$ set of infinite words
- $u^\omega = uuu\cdots$, with $u \in \mathcal{A}^+$
- Given a total order $<$ on $\mathcal{A}$, we define the lexicographical order on $\mathcal{A}^\omega$ as

$$\mathbf{u} < \mathbf{v} \quad \Longleftrightarrow \quad \mathbf{u} = pa\mathbf{s},\ \mathbf{v} = pb\mathbf{t} \quad p \in \mathcal{A}^*,\ a, b \in \mathcal{A},\ a < b,\ \mathbf{s}, \mathbf{t} \in \mathcal{A}^\omega$$

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When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

Example

$\mathtt{ab} < \mathtt{aba}$ but $(\mathtt{aba})^\omega < (\mathtt{ab})^\omega$.

$u^\omega = v^\omega \iff u$ and v are power of a common word ($\iff uv = vu$).

Infinite words

Theorem [Fine & Wilf] (More in the next talk)

Let $u, v \in \mathcal{A}^+$ be s.t. $u^\omega \neq v^\omega$.
Then, the first mismatch is at position $k \leq |u| + |v| - \gcd(|u|, |v|)$.

Infinite words

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Example

$u = \text{abaab}, \qquad v = \text{abaababa}.$
The first mismatch is at position $12 = |u| + |v| - 1$.

Infinite words

Proposition

Let $u, v \in \mathcal{A}^+$ be s.t. $u^\omega \neq v^\omega$ and let $\mathbf{s}, \mathbf{t} \in \{u, v\}^\omega$.
Then, $u^\omega < v^\omega \iff u\mathbf{s} < v\mathbf{t}$.

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Theorem [D., Restivo, Reutenauer (2018)]

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega < v^\omega$,

(2) $(uv)^\omega < v^\omega$,

(3) $u^\omega < (vu)^\omega$,

(4) $(uv)^\omega < (vu)^\omega$,

(5) $u^\omega < (uv)^\omega$,

(6) $(vu)^\omega < v^\omega$.

Důkaz na tabuli.

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Důkaz na tabuli.

Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega < (uv)^\omega < (vu)^\omega < v^\omega$.

Lyndon words and infinite words

Corollary

Let $u, v \in \mathcal{A}^+$. Then $u^\omega < v^\omega$ iff $uv < vu$.

Lyndon words and infinite words

Corollary

Let $u, v \in \mathcal{A}^+$. Then $u^\omega < v^\omega$ iff $uv < vu$.

Theorem [D., Restivo, Reutenauer (2018)]

A word w is Lyndon iff for any non-trivial factorization $w = ps$ one of the following equivalent conditions is satisfied:

(1) $p^\omega < s^\omega$,

(2) $(ps)^\omega < s^\omega$,

(3) $p^\omega < (sp)^\omega$,

(4) $(ps)^\omega < (sp)^\omega$,

(5) $p^\omega < (ps)^\omega$ [Ufnarovskij (1995)],

(6) $(sp)^\omega < s^\omega$.

Factorization into classical Lyndon words

Theorem [Ufnarovskij (1995)]

Let $w = \ell_1 \ell_2 \cdots \ell_n$ the unique non-increasing factorization of w in Lyndon word. Then

- $\ell_1^\omega > (\ell_2 \cdots \ell_n)^\omega$
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq s^\omega$,
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Example

$\mathtt{w} = \mathtt{aab.aab.a}$

- $(\mathtt{aab})^\omega > (\mathtt{aab.a})^\omega$,
- $(\mathtt{aab})^\omega > (\mathtt{aab.aab.a})^\omega$.

Complete trees

The set of *complete trees* over $\mathcal{A}$ is defined recursively as follows:

- each letter $a \in \mathcal{A}$ is a tree;
- if $\mathfrak{t}_1, \mathfrak{t}_2$ are trees, then $(\mathfrak{t}_1, \mathfrak{t}_2)$ is a tree.

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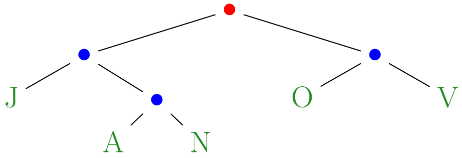

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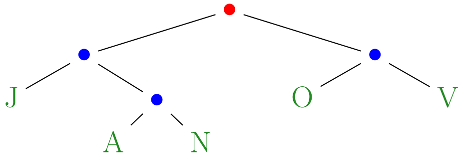

We will use the classical notions of *root*, *internal node* and *leaf* for a tree.
The *foliage* $\varphi(\mathfrak{t})$ of a tree $\mathfrak{t}$ is defined as:

- $\varphi(a) = a$ for any $a \in \mathcal{A}$,
- $\varphi((\mathfrak{t}_1, \mathfrak{t}_2)) = \varphi(\mathfrak{t}_1)\varphi(\mathfrak{t}_2)$ for any two trees $\mathfrak{t}_1, \mathfrak{t}_2$.

Left standard factorization

Definition

Let w be a Lyndon word of length at least 2.
The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Example

The left standard factorization of aabaacab is aabaac.ab.

Left standard factorization

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Proposition

Both u and v are Lyndon words.
Moreover, either v is a letter or $v = v_1 v_2$, and $v_1 \leq u$.

Example

The left standard factorization of $\mathtt{aabaacab}$ is $\mathtt{aabaac.ab}$.
The left standard factorization of $\mathtt{ab}$ is $\mathtt{a.b}$, and $\mathtt{a} \leq \mathtt{aabaac}$.

Left Lyndon tree

Let $w \in \mathcal{A}^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as:

- $\mathcal{L}(a) = a$ for each letter $a \in \mathcal{A}$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.

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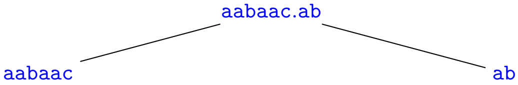

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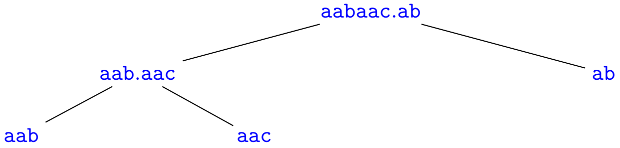

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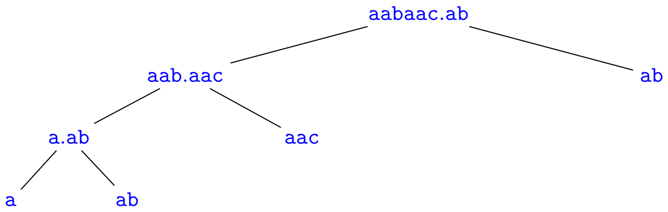

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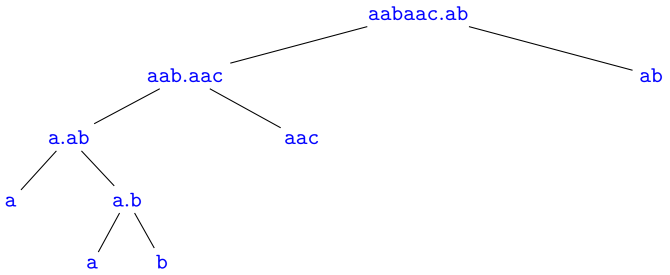

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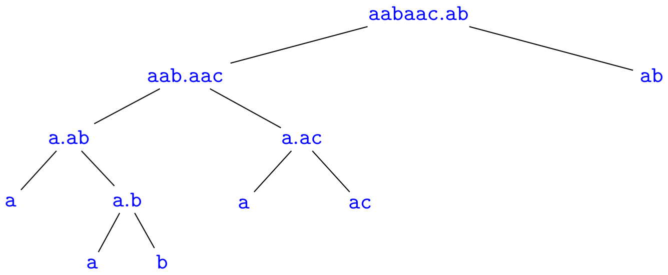

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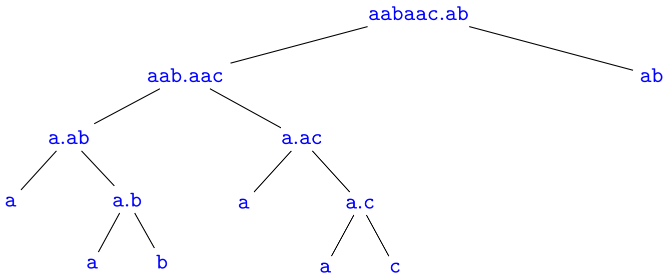

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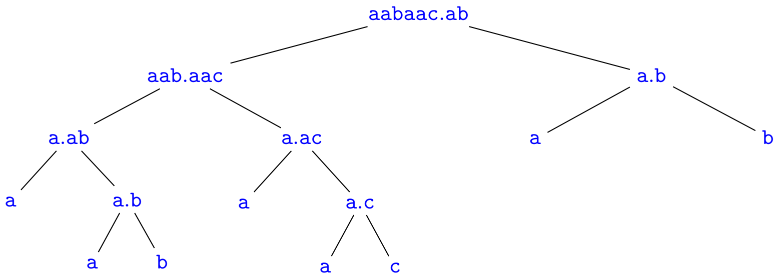

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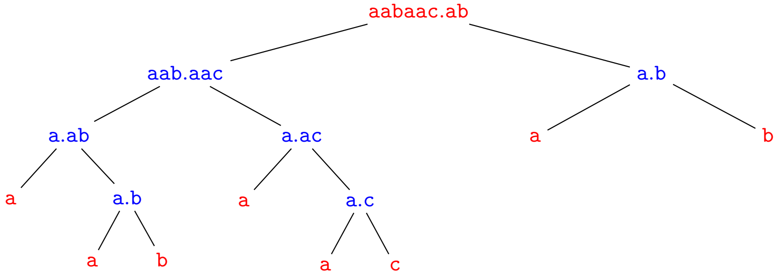

Clearly $\varphi(\mathcal{L}(w)) = w$.

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$\mathtt{aa} \prec \mathtt{a} \prec \mathtt{ab} \prec \mathtt{ba} \prec \mathtt{b}$.

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$$w = \quad \texttt{aabaacab}$$

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1

aa

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Example

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$$21543$$

$\mathtt{aa} \prec \mathtt{a} \prec \mathtt{aabaa} \prec \mathtt{aaba} \prec \mathtt{aab}$

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21543 6

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Example

$w =$ aabaacab

2154376

aa $\prec$ a $\prec$ aabaa $\prec$ aaba $\prec$ aab $\prec$ aabaaca $\prec$ aabac

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Example

$$w = \begin{matrix} \mathtt{aabaacab} \\ 21543768 \end{matrix}$$

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$$\begin{array}{rl} w = & \mathtt{aabaacab} \\ & 21543768 \end{array}$$

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aabaacab $\longleftrightarrow$ 2154376

7

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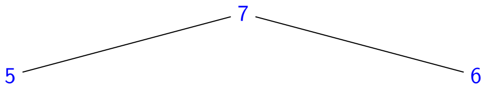
`aabaacab` $\longleftrightarrow$ 2154376



Left Cartesian tree

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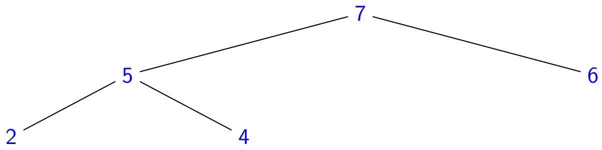

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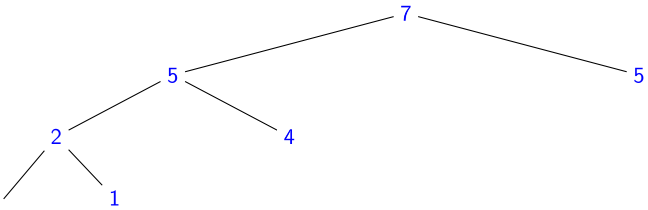

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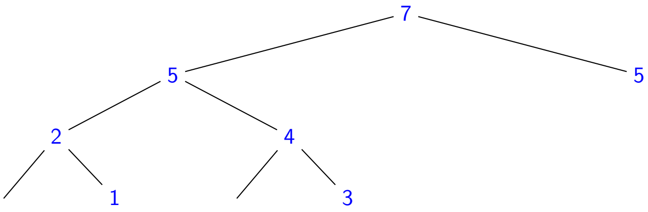

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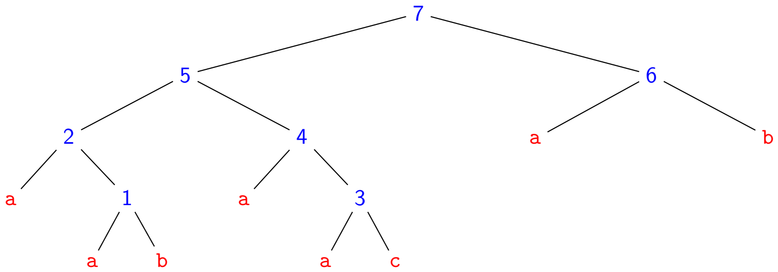


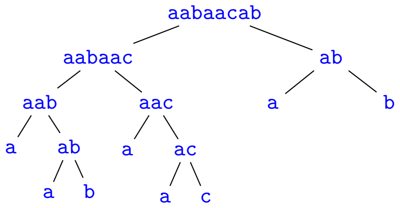
We complete in such a way that $\varphi(\mathcal{C}(w)) = w$.

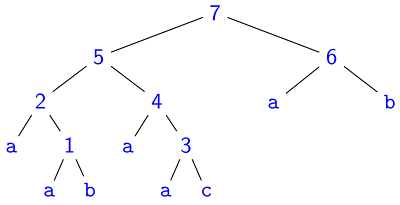
Equivalence of trees

Theorem [D., Restivo, Reutenauer (2019)]

Let w be a Lyndon word. Then $\mathcal{L}(w) = \mathcal{C}(w)$.

aabaacab $\longleftrightarrow$ 2154376






Mmmmoc děkuji
za pozornost