Specular sets

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WORDS

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Joint work with

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Introduction

Generalization of links between Sturmian sets and Free groups to general objects : *Specular sets* and *Specular groups*.

Introduction of new concepts : *parity* of words (*odd* and *even* words), *mixed return words*.

Framework allowing to handle linear involutions (generalization of interval exchanges).

Adaptation of results holding for tree sets : *Maximal Bifix Decoding Theorem*, *Finite Index Basis Theorem*, *Return Theorem*.

Outline

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Given an involution $\theta : A \to A$ (possibly with some fixed point), let us define

$$G_\theta = \langle a \in A \mid a \cdot \theta(a) = 1 \text{ for every } a \in A \rangle.$$

$G_\theta = \mathbb{Z}^i * (\mathbb{Z}/2\mathbb{Z})^j$ is a *specular group* of type (i, j), and $\mathrm{Card}\,(A) = 2i + j$ is its *symmetric rank*.

Example

Let $A = \{a, b, c, d\}$ and let θ be the involution which exchanges b, d and fixes a, c, i.e.,

$$G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = db = 1 \rangle.$$

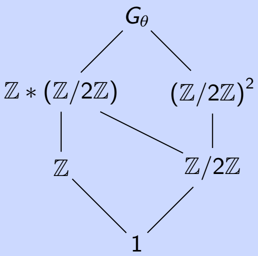
$G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$ is a specular group of type $(1, 2)$ and symmetric rank 4.

Theorem

Any subgroup of a specular group is specular.

Example

Let $G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$, then one has



A word is θ-*reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Any element of a specular group is represented by a unique reduced word.

Example

Let θ be the involution on the alphabet $\{a, b, c, d\}$ that fixes a, c and exchanges b, d.

The θ-reduction of the word $daaacbd$ is dac.

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The θ-reduction of the word $d\not{a}\not{a}ac\not{b}\not{d}$ is dac.

A subset of a group G is called *symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

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Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

A set X in a specular group G is called a *monoidal basis* of G if :

- it is symmetric ;
- the monoid that it generates is G ;
- any product $x_1 x_2 \cdots x_m$ such that $x_k x_{k+1} \neq 1$ for every k is distinct of 1.

Example

The alphabet A is a monoidal basis of G_θ.

The *symmetric rank* of a specular group is the cardinality of any monoidal basis.

Outline

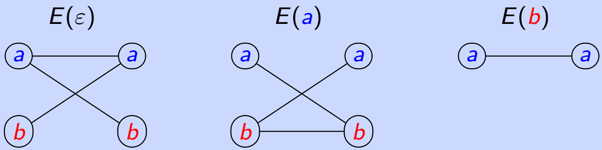
Let S be a factorial over an alphabet A.
The *extension graph* of a word $w \in S$ is the undirected bipartite graph $G(w)$ with vertices the disjoint union of

$$L(w) = \{a \in A \mid aw \in S\} \quad \text{and} \quad R(w) = \{a \in A \mid wa \in S\},$$

and edges the pairs $E(w) = \{(a, b) \in A \times A \mid awb \in S\}$.

Example

The *Fibonacci set* is the set of factors of the Fibonacci word, i.e. the fixed point $\varphi^\omega(a)$ of the morphism $\varphi : a \mapsto ab, b \mapsto a$.



Indeed one has $S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.

A biextendable set S is called a *tree set* of *characteristic* c if for any nonempty $w \in S$, the graph $E(w)$ is a tree (acyclic and connected) and if $E(\varepsilon)$ is a union of c trees.

Example

The Fibonacci set is a tree set of characteristic 1.

A biextendable set S is called a *tree set* of *characteristic* c if for any nonempty $w \in S$, the graph $E(w)$ is a tree (acyclic and connected) and if $E(\varepsilon)$ is a union of c trees.

Example

The Fibonacci set is a tree set of characteristic 1.

Proposition [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

Factors of an Arnoux-Rauzy word and regular interval exchange sets are both uniformly recurrent tree sets of characteristic 1.

Example

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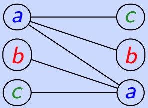
$E(\varepsilon)$



A *specular set* on an alphabet A (w.r.t. an involution θ) is a

- biextendable and
- symmetric set
- of θ-reduced words
- which is a tree set of characteristic 2.

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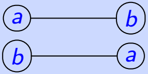
Example

Let $A = \{a, b\}$ and θ be the identity on A. The set of factors of $(ab)^\omega$ is a specular set.

$E(\varepsilon)$



Proposition [J. Cassaigne (1997)]

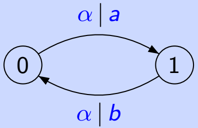
The factor complexity of a specular set is given by $p_0 = 1$ and $p_n = n\,(\mathrm{Card}\,(A) - 2) + 2$.

A *doubling transducer* is a transducer with set of states $Q = \{0, 1\}$ on the input alphabet Σ and the output alphabet A such that :

1. the input automaton is a group automaton, that is, every letter of Σ acts on Q as a permutation,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



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A *doubling map* is a pair $\delta = (\delta_0, \delta_1)$, where $\delta_0, \delta_1 : \Sigma^* \to A^*$ are two maps such that $\delta_i(u) = v$ is the path starting at the state i with input label u and output label v.

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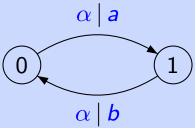

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The *image* of a set T by a doubling map is the set $\delta(T) = \delta_0(T) \cup \delta_1(T)$.

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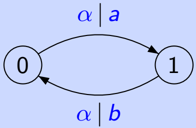

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$\delta(\alpha^\omega) = (ab)^\omega \cup (ba)^\omega$

Proposition

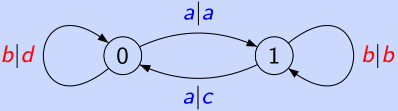
The image of a tree set of characteristic 1 closed under reversal by a doubling map is a specular set.

Example

Two possible *doublings* of the Fibonacci set are :

- the set of factors of the two infinite sequences $abaababa\cdots$ and $cdccdcdc\cdots$,

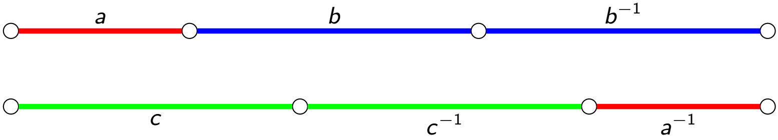
- the set of factors of the two infinite sequences $abcabcda\cdots$ and $cdacdabc\cdots$.



Both are specular sets. Their factor complexity is $2n+2$.

Theorem

The natural coding of a linear involution without connections is a specular set.



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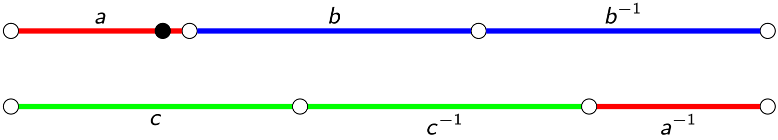

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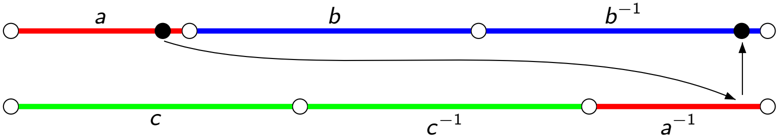

$$\Sigma_T(z) = a\, b^{-1}$$

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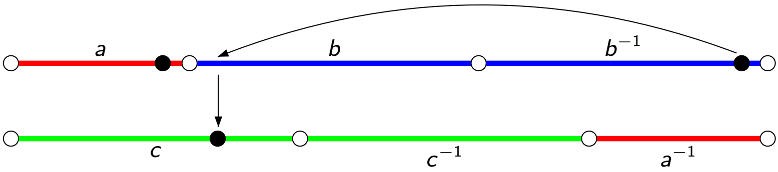

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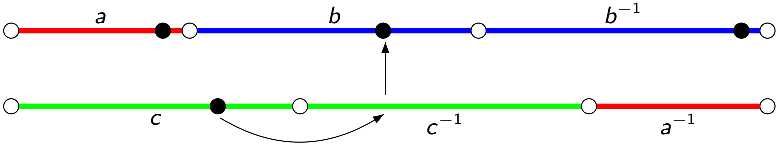

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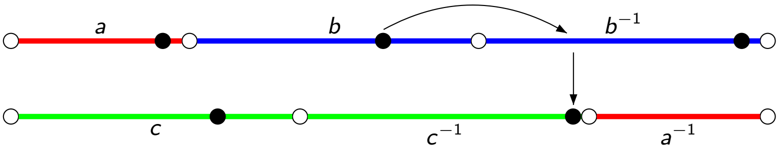
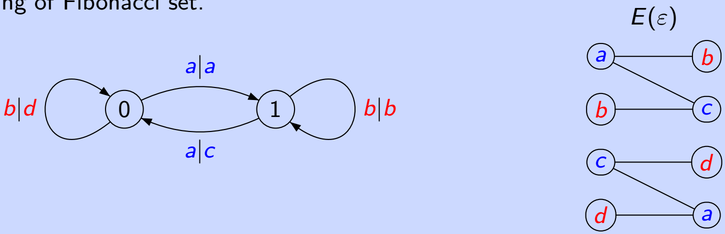


$$\Sigma_T(z) = a\, b^{-1}\, c\, b\, c^{-1} \cdots$$

A letter is said to be *even* if its two occurences (as a element of $L(\varepsilon)$ and of $R(\varepsilon)$) appear in the same tree of $E(\varepsilon)$. Otherwise it is said to be *odd*.

Example

Doubling of Fibonacci set.



The letters b, d are even, while the letters a, c are odd.

A word is said to be *even* if it has an even number of odd letters. Otherwise it is said to be *odd*.

Outline

A set $X \subset A^+$ of nonempty words over an alphabet A is a *bifix code* if it does not contain any proper prefix or suffix of its elements.

Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
- $\{ac, bcc, bcbca\}$

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A bifix code $X \subset S$ is *S-maximal* if it is not properly contained in a bifix code $Y \subset S$.

Example

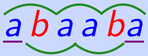
Let S be the Fibonacci set. The set $X = \{aa, ab, ba\}$ is an S-maximal bifix code. It is not an A^*-maximal bifix code, indeed $X \subset Y = X \cup \{bb\}$.

A *parse* of a word w with respect to a bifix code X is a triple (q, x, p) with $w = qxp$ and such that q has no suffix in X, $x \in X^*$ and p has no prefix in X.

Example

Let $X = \{aa, ab, ba\}$ and $w = abaaba$. The two possible parses of w are

- $(\varepsilon, ab\,aa\,ba, \varepsilon)$,
- $(a, ba\,ab, a)$.



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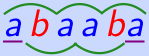


The S-*degree* of X is the maximal number of parses with respect to X of a word of S.

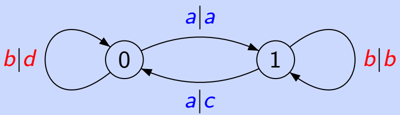
Example

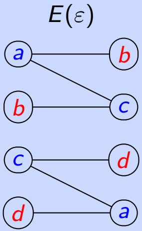
- For the Fibonacci set S, the set $X = \{aa, ab, ba\}$ has S-degree 2
- The set $X = S \cap A^n$ has S-degree n.

The set of even words in a specular set S has the form $X^* \cap S$, where $X \subset S$ is a bifix code called the *even code*.
The set X is the set of even words without a nonempty even prefix (or suffix).

Example

Doubling of Fibonacci set.





The even code is $X = \{abc, ac, b, ca, cda, d\}$.

Proposition

The even code of a recurrent specular set S is an S-maximal bifix code of S-degree 2.

Let S be a factorial set and X be a finite S-maximal bifix code.
A *coding morphism* for X is a morphism $f : B^* \to A^*$ which maps bijectively an alphabet B onto X.

The set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

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Maximal Bifix Decoding Theorem

The decoding of a uniformly recurrent specular set by the even code is a union of two uniformly recurrent tree sets of characteristic 1.

Example

The set $S = \text{Fac}\left((ab)^\omega\right)$ is a specular set. Its even code is $X = \{ab, ba\}$.
Let us consider the coding morphism for X

$$f : \begin{cases} u \mapsto ab \\ v \mapsto ba \end{cases}$$

Then, $f^{-1}(S) = \text{Fac}\left(u^\omega\right) \cup \text{Fac}\left(v^\omega\right)$.

Finite Index Basis Theorem

Let S be a uniformly recurrent specular set and $X \subset S$ a finite symmetric bifix code. X is an S-maximal bifix code of S-degree d if and only if it is a monoidal basis of a subgroup of index d.

Example

- $S \cap A^n$.
- The even code is a monoidal basis of a subgroup of index 2 of G_θ called the *even subgroup*.

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- The even code is a monoidal basis of a subgroup of index 2 of G_θ called the *even subgroup*.

The Finite Index Basis Theorem has also a converse.

Theorem

Let S be a recurrent and symmetric set of reduced words having factor complexity $p_n = n\,(\mathrm{Card}\,(A) - 2) + 2$.
If $S \cap A^n$ is a monoidal basis of the subgroup $\langle A^n \rangle$ for all $n \geq 1$, then S is a specular set.

Let S be a factorial set of words and $x \in S$.
A (*right*) *return word* to x in S is a nonempty word u such that $xu \in S \cap A^* x$, but has no internal factor equal to x.

We denote by $\mathcal{R}_S(w)$ the set of return words to x in S.

Example

Let S be the Fibonacci set. One has $\mathcal{R}_S(aa) = \{baa, babaa\}$.

$$\varphi(a)^\omega = ab\underline{aa}bab\underline{aa}baababaababaab\underline{aa}babaabaab\cdots$$

Remark. A recurrent set S is uniformly recurrent if and only if the set $\mathcal{R}_S(w)$ is finite for every $w \in S$.

Theorem [Balková, Palentová, Steiner (2008)]

Let S be a uniformly recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}_S(w)$ has exactly $\text{Card}\,(A)$ elements.

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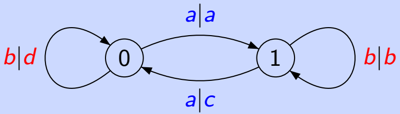
Return Theorem

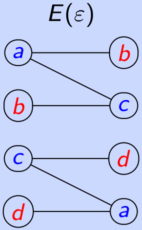
Let S be a uniformly recurrent specular set on the alphabet A.
For any $w \in S$, the set $\mathcal{R}_S(w)$ is a basis of the even subgroup.

In particular, $\mathrm{Card}\,(\mathcal{R}_S(x)) = \mathrm{Card}\,(A) - 1$.

Example

Let $G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = 1 \rangle$ and S be the doubling of the Fibonacci set :





The even code is $X = \{abc, ac, b, ca, cda, d\}$,
while $\mathcal{R}_S(a) = \{bca, bcda, cda\}$.

Then, $\langle \mathcal{R}_S(a) \rangle = \langle X \rangle$, indeed :

$$\begin{cases} cda = cda & ca = (b)^{-1}(bca) \\ abc = (cda)^{-1} & ac = (ca)^{-1} \\ b = (bcda)(abc) & d = b^{-1} \end{cases}$$

Conclusions

Quick summary for those who fell asleep (wake up : it's lunch time!)

- Introduction of specular groups and specular sets.
- Generalization within these sets of results holding for tree sets.

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Further research directions

- Investigation about recurrence (uniformly recurrence and tree condition, bifix decoding, . . .).
- Interesting connection with G-full (or G-rich) words.
- Generalization towards larger classes of groups (virtually free).


Danke !