Knots, knots. Who's there?

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1 Introduction and definitions

When Alexander the Great entered the city of Gordion, the oracle told him of the ancient prophecy: whoever would first untie the sacred knot would became the ruler of all Asia. Many people before struggled to unravel the knot, all without success. Alexander stopped to think for a moment. Then he drew his sword and with a single stroke cut the knot in half. Later, the great Macedonian king went on to conquer Asia as far as the Indus and the Oxus, thus fulfilling the prophecy. From a mathematical point of view, Alexander cheated and its solution cannot be considered a valid one. But myth and math do not always go hand in hand.

Figure 1: Details of a knot on a wall in Prague.

But what is a knot for a mathematician? Knot theory can be seen as a branch of topology (even though we can study knots with tools from different branches of mathematics). A knot is a closed curve in a 3-dimensional space $\mathbb{R}^3$, like a rope in our usual three-dimensional world that has first been tangled up and whose extremities have then been glued together. The simplest knot it the circle itself, denoted as $\mathbb{S}^1$. Because of its simplicity it is called the *unknot* (see, for instance, the left of Figure 2).

If we want to be more formal, we can use the notion of *homeomorphism*, that is a continuous function (a fuction that sends neighboring points to neighboring points) having a continuous inverse[1]. With such a tool we can easily pass from one closed curve (the twisted rope we discussed before) to another one, and also coming back, avoiding weird and unpleasant situations.

Definition 1. *A* knot *is a subspace* $K \subset \mathbb{R}^3$ *such that* $\varphi(K) = \mathbb{S}^1$ *for some homeomorphism* $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$.

For example, the *trefoil knot* represented on center of Figure 2 is a knot.

When we want to study several knots at the same time, the following definition can come in handy.

Definition 2. *A* link L *is a union* $L = K_1 \cup K_2 \cup \cdots \cup K_m$ *of* m *knots* K_i, *with* $m \in \mathbb{N}$ *and* $1 \leq i \leq m$, *such that* $K_j \cap K_k = \emptyset$ *for every* $j \neq k$.

A romantic example of a link is the *Hopf link*, a union of two unknot, represented on the right of Figure 2.

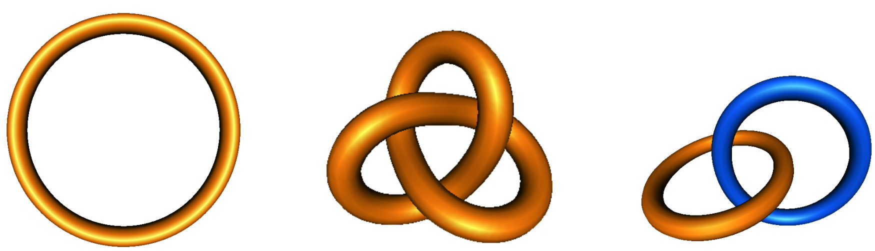


Figure 2: The unknot (left), the trefoil knot (center) and the Hopf link (right).

An *oriented knot* (or an oriented component of a link) is just a knot on which we have defined a preferred direction. An example of an oriented knot (the so-called *figure-eight knot*) and of an oriented link are shown in Figure 3.

2 Projections

Since the knots live in a 3-dimensional space, but on a sheet of paper we only can use two of these dimensions, a useful tool to study knots is the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, defined as $\pi(x, y, z) = (x, y)$.

[1] A continuous inverse is needed because there can be different topologies in $\mathbb{R}^3$, i.e., points that are neighboring in one topology do not have to be close in some other topology. For precise defintions of topology and homeomorphism see [6].

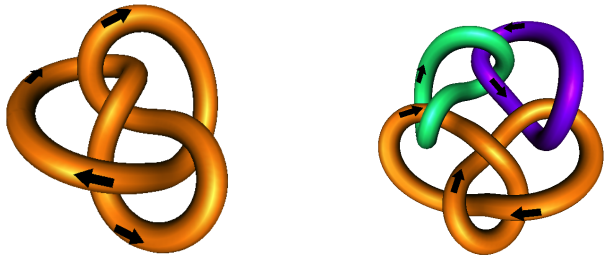


Figure 3: An oriented knot (left) and an oriented link (right).

Definition 3. *Given a knot* $K \subset \mathbb{R}^3$*, we call the* projection *of* K *the* 2-*dimensional closed line* $\pi(K) \subset \mathbb{R}^2$.

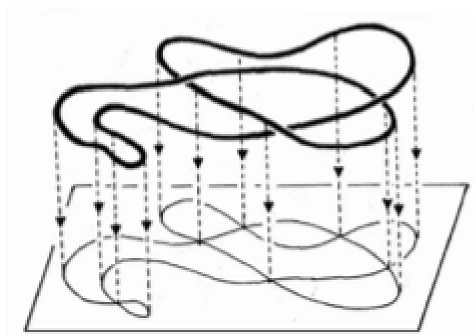
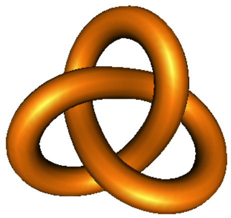


Figure 4: Projection of a knot.

A point $P \in \pi(K)$ is called a *multiple point* if $\pi^{-1}(P)$ contains at least 2 points. In our projections we suppose that the only multiple points have order exactly 2, that is they correspond to only two points in the original knot (if this is not the case, we can always "move" the knot a little bit in order to obtain a projection with such property). We will call such double points *intersection* of the projection. We also denote as *upper branch* and *lower branch* of the intersection P respectively the neighborhood of the point in $\pi^{-1}(P)$ with bigger and with smaller z coordinate. If two intersections are both of un upper branch or both on a lower branch, we say that they are of the same type. To distinguish the two branches, we usually represent the upper branch with a continuous line and the lower branch with a discontinuous line at the intersection, as shown in Figure 5.



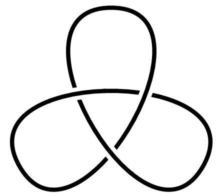


Figure 5: The trefoil knot (left) and one of its projections (right).

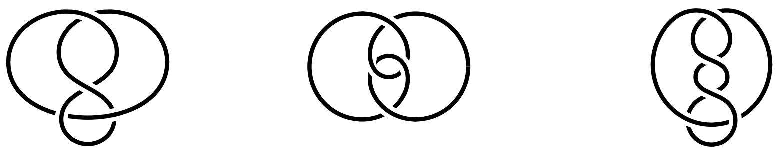


Figure 6: Three equivalent knots.

3 Equivalent knots

One of the main questions in the mathematical theory of knots is to determine whenever two knots are actually "the same knot" but represented in a different way. Intuitively that means that we can pass from the first knot to the second one by a series of small continuous movements, without using a pair of scissors or some glue (preserving the orientation in the case of oriented knots). The following is a formalisation of this notion.

Definition 4. *Two knots K_1 and K_2 are* equivalent *if there exists an* ambient isotopy *between the two nodes, that is if there exists a continuous map $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that*

i) for every $t \in [0, 1]$, the map $h_t : x \to h(x, t)$ is a homeomorphism (in particular h_t sends the circle to a certain knot);

ii) the map h_0 sends the circle to the first knot K_1;

iii) the map h_1 sends the circle to the second knot K_2.

An example of three equivalent knots is given in Figure 6.

The problem with Definition 4 is that in order to check the equivalence of two knots we need an infinite number of intermediary homeomorphisms (or an infinite number of "small movements"). Even in the apparently easier case when one of the two knots is the unknot. For instance, the knots in Figure 7, known respectively as the *Goeritz's unknot* and the *Haken's unknot* are, despite their appareance, equivalent to the circle.

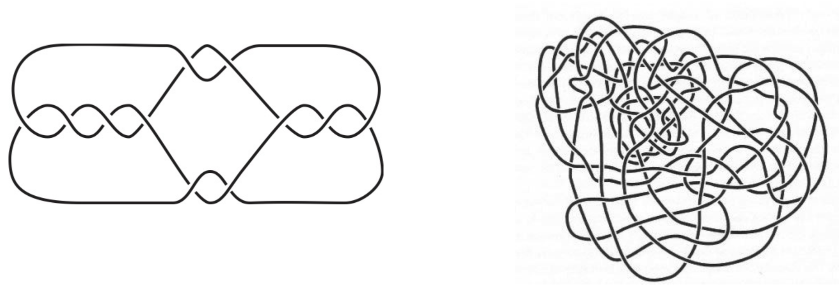


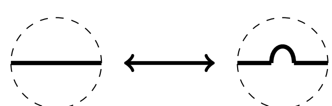
Figure 7: Goeritz' unknot (on the left) and Haken's unknot (on the right).

Wolfgang Haken proposed in 1961 one of the first unknot recognition algorithms. However, such an algorithm has very big complexity, so it cannot be efficiently implemented. Recently Marc Lackenby proposed an algorithm that, given a diagram of a knot with n crossings, determine in quasi-polynomial time (more precisely in $n^{\mathcal{O}(\log n)}$ time), whether this is the unknot or not. It is still unknown if this is the best possible solution or if a more efficient algorithm can be found (you should try!).

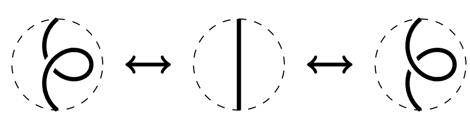
4 Reidemeister moves

In 1932 Kurt Reidemeister proved that two 2-dimensional closed lines represent the same knot if and only if it is possible to pass from the first one to the second one through a succession of *moves*, or local changes of the following types:

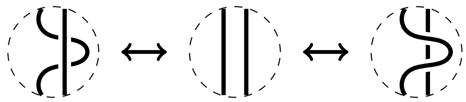
- planar isotopy (or Ω_0) not creating or destroying any intersections;



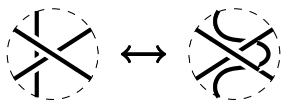
- Ω_1, creating or removing an intersection relative to a loop;

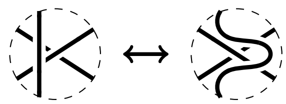


- Ω_2, creating or removing two successive intersections of the same type by moving a strand over another;



- Ω_3, moving two consecutive intersections of the same type over a third intersection.





Theorem 5 (Reidemeister)**.** *Two knots are equivalent if and only if it is possible to obtain a projection of the second knot from a projection of the first knot by a sequence of Reidemeister moves.*

It is important to note that the previous theorem does not provide any algorithm to establish whether two knots are equivalent to each other. Indeed, we do not know which sequence of Reidemeister moves we need to use to pass from one projection to the other. In general, it is not even guaranteed that the right sequence of moves will simplify the diagram, that is, reduce the number of intersections.

5 Mirror image and opposite

How can we use known knots to construct new ones? A possible way is taking a knot and either considering its mirror image or changing its orientation.

Definition 6. *The* mirror image *of a knot K is the knot obtained starting from a projection of K by simply inverting the upper and lower branches at each intersection.*

A knot that is equivalent to its mirror image is called *achiral*. For instance, the trefoil knot is not achiral, while the figure-eight knot is achiral (see Figure 8).

Definition 7. *The* opposite *knot of an oriented knot is the oriented knot obtained just by changing the orientation.*

As for the mirror image, some knots are equivalent to their opposite and some are not (see, e.g., Figure 9).

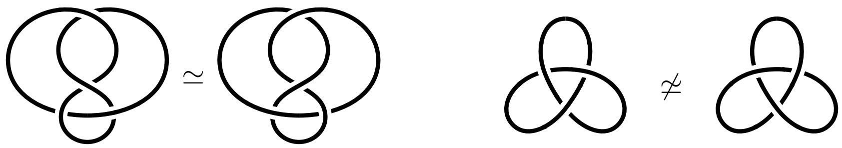


Figure 8: The figure-eigth knot (on the left) is chiral, while the trefoil knot (on the right) is not.

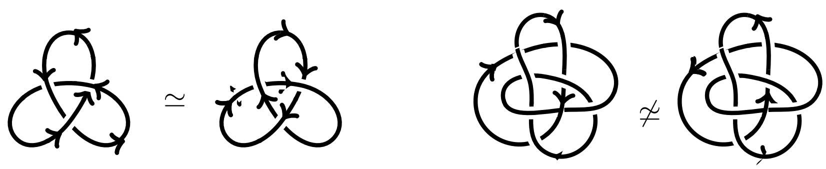


Figure 9: The trefoil knot (on the left) is equivalent to its opposite while the knot 8_{17} (on the right) is not.

6 Knot sum

Another way of obtaining a knot is by *connected sum* of two knots.

Definition 8. *The* connected sum *of two knots J and K is the knot $J \# K$ obtained by cutting both knots and joining the pairs of ends without creating new intersactions.*

In the knots are oriented, we should be careful to connect them in such a way that the direction is consistent. An example of sum of two knots is given in Figure 10.

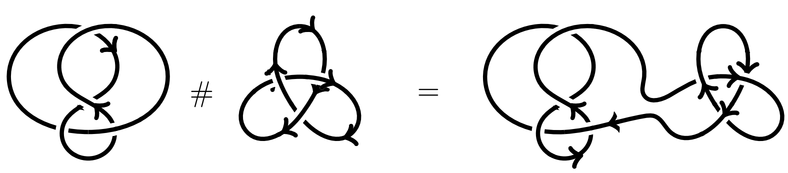


Figure 10: The connected sum of two knots.

It is easy to see that such operation is both commutative and associative. Moreover, the unknot is clearly the neutral element of this operation.

Similarly to integers, we say that a knot is a *prime knot* if it is not possible to obtain it as a sum of two non-trivial knots. A knot that is not prime is called a *composite knot*. As for integer numbers, we also have a fundamental theorem for knots, proved by Herbert Seifert in 1949.

Theorem 9 (Seifert)**.** *Every knot can be factorized in a unique way as a connected sum of prime knots.*

The study of knots is not only elegant and gives the opportunity to have fun drawing, but has also several applications in different aspects of everyday life (have you ever noticed how complicated it is to untangle your headphone cables?) as well as in science, from molecular biology to quantum physics.

References

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