Maximal bifix decoding of a tree set

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Automatic Sequences
Liège, 25th May 2015

Joint work with
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Motivation

$$x = abaababaabaababa\cdots$$

$$x = \varphi^{\omega}(a)$$

$$\varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

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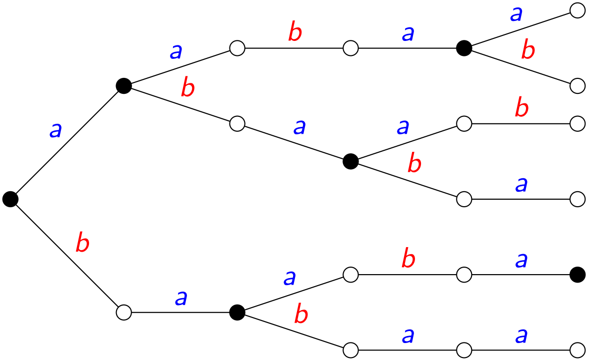


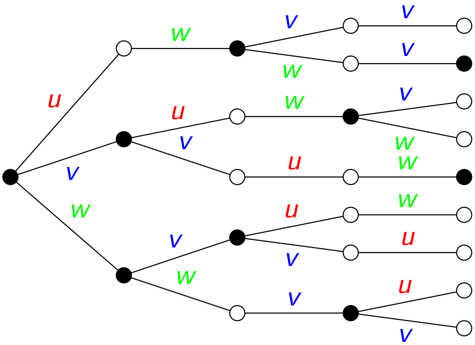
n	0	1	2	3	4	5	...
(2-1)n+1	1	2	3	4	5	6	...

Motivation

$$x = \underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ba}\,\underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ba}\cdots$$

$$f(x) = v\ u\ w\ w\ v\ u\ w\ w\cdots$$

$$f : \begin{cases} u = aa \\ v = ab \\ w = ba \end{cases}$$



n	0	1	2	3	4	$\cdots$
(3-1)n+1	1	3	5	7	9	$\cdots$

Outline

Motivation

1. Two important classes
2. Acyclic, connected and tree sets
3. Maximal bifix decoding

Outline

Motivation

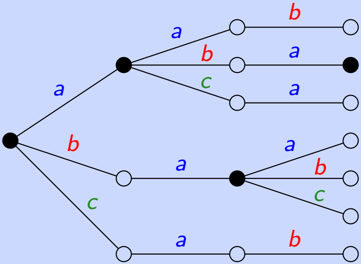
1. Two important classes
    - Sturmian sets
    - Interval Exchange sets
2. Acyclic, connected and tree sets
3. Bifix decoding

A *Sturmian* set is the set of factors of a *strict episturmian word* (i.e. of a word x whose set of factors $F(x)$ is closed under reversal and for each n contains exactly one right-special word w_n of length n with $w_n A \subset F(x)$).

Example

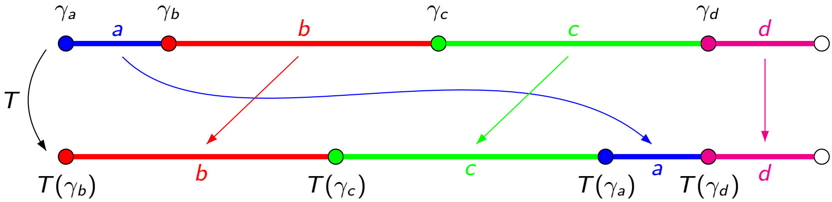
Let $A = \{a, b, c\}$. The *Tribonacci set* is the set of factors of the Tribonacci word, i.e. the fixed point $x = \psi^\omega(a) = abacaba \cdots$ of the morphism

$$\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$$



Let A be a finite set ordered by $<_1$ and $<_2$.
An *interval exchange transformation* (IET) is a map $T : [0, 1[\to [0, 1[$ defined by

$$T(z) = z + \alpha_z \quad \text{if } z \in I_a.$$



$a <_1 b <_1 c <_1 d$ $\qquad$ $b <_2 c <_2 a <_2 d$

An interval exchange transformation T is said to be *minimal* if for any $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

The transformation T is said *regular* if the orbits of the nonzero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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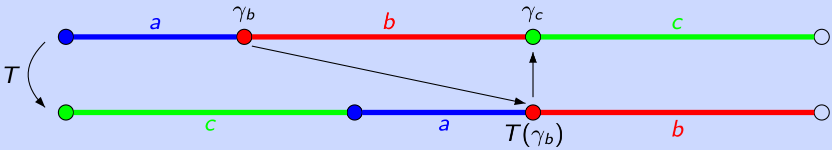
Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

The converse is not true.

Example

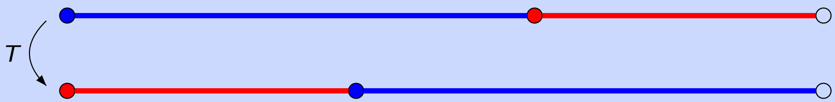


Let T be an IET relative to $(I_a)_{a \in A}$.
The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in I_a.$$

Example

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3-\sqrt{5})/2$ relative to the point α, i.e. $T(z) = z + \alpha \bmod 1$.



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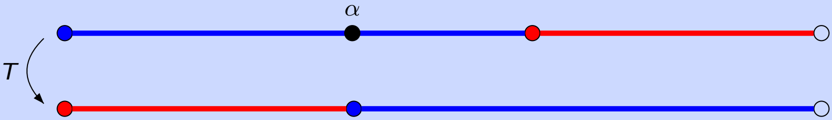

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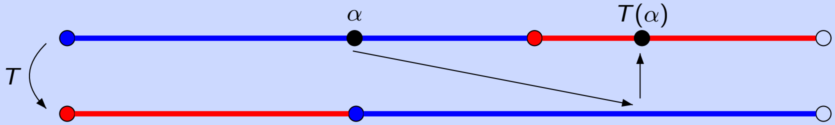

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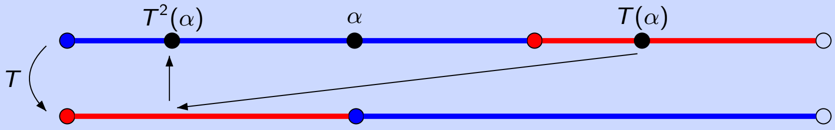

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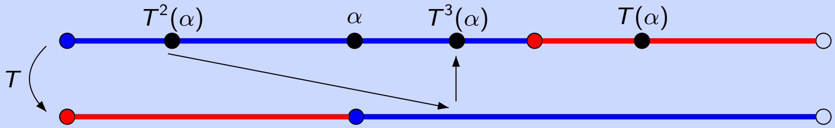

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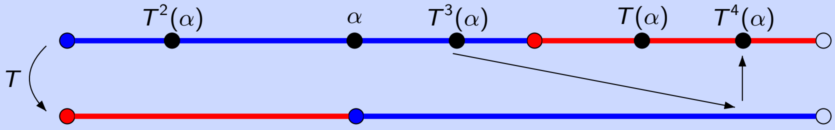

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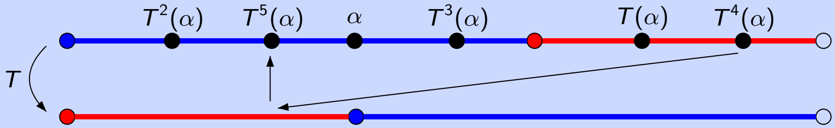


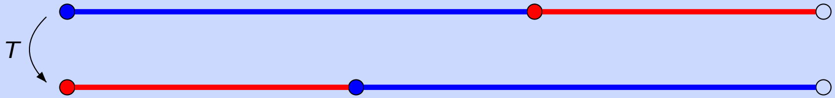
$$\Sigma_T(\alpha) = a\,b\,a\,a\,b\,a \cdots$$

The set $F(T) = \bigcup_{z\in[0,1[} (\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $F(\Sigma_T(z))$ does not depend on the point z.

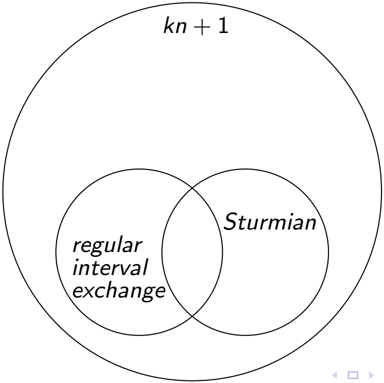
Example

The *Fibonacci set* is the set of factors of a natural coding of the rotation of angle $\alpha = (3-\sqrt{5})/2$.



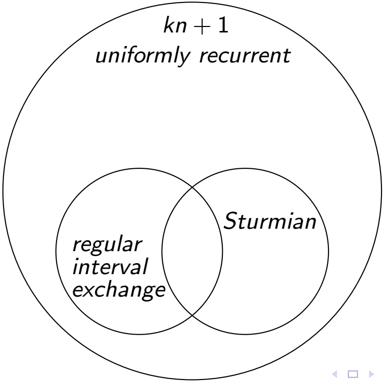
$$F(T) = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, \dots\}$$

Sturmian sets and regular interval exchange sets have both complexity function $p(n) = kn + 1$, with $k = \text{Card}(A) - 1$.



Sturmian sets and regular interval exchange sets have both complexity function $p(n) = kn + 1$, with $k = \text{Card}\,(A) - 1$.

They are factorial and *uniformly recurrent* (right-extendable and s.t. for any element $u \in S$ there exists an $n = n(u)$ with u a factor of all words of $S \cap A^n$).



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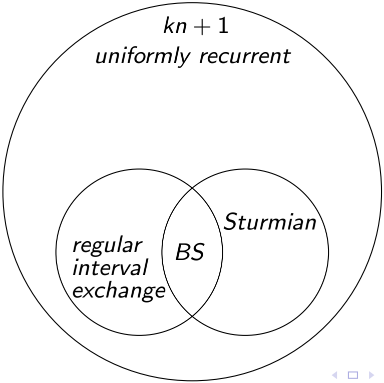
However, the two families are distinct for $k \geq 2$.



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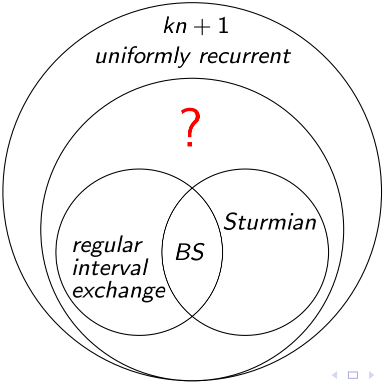
However, the two families are distinct for $k \geq 2$.

Do they have other properties in common ?



Outline

Motivation

1. Two important classes
2. Acyclic, connected and tree sets
   - Tree sets
   - Planar tree sets
3. Bifix decoding

Let S be a factorial set over an alphabet A.
The *extension graph* of a word $w \in S$ is the undirected bipartite graph $G(w)$ with vertices the disjoint union of

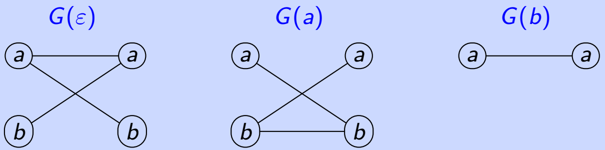
$$L(w) = \{a \in A \,|\, aw \in S\} \quad \text{and} \quad R(w) = \{a \in A \,|\, wa \in S\},$$

and edges the pairs in

$$E(w) = \{(a, b) \in A \times A \,|\, awb \in S\}.$$

Example

Let S be the Fibonacci set.



Indeed one has $S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.

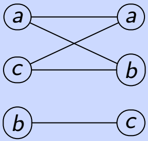
A set S is *acyclic* (resp. *connected*) if it is biextendable and if for every word $w \in S$, the graph $G(w)$ is acyclic (resp. connected).

A set S is a *tree set*[1] if $G(w)$ is acyclic and connected for every word $w \in S$.

Example

Let $A = \{a, b, c\}$. The set S of factors of $a^*(bc + bcbc)a^*$ is not a tree set. Actually it is neither acyclic nor connected.

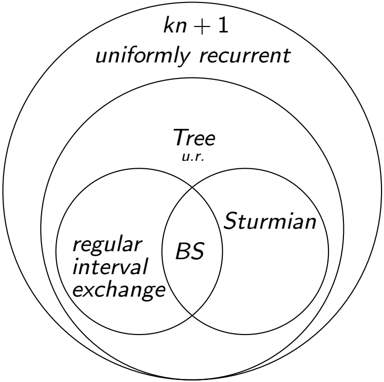
$G(\varepsilon)$



1. of characteristic $\chi(S) = 1$.

Proposition [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

Both Sturmian sets and regular interval exchange sets are uniformly recurrent tree sets.

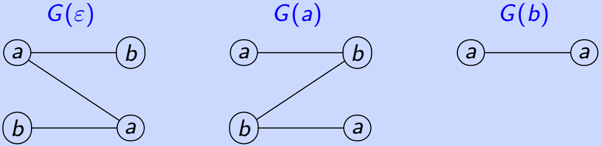


Let $<_1$ and $<_2$ be two orders on A.
For a set S and a word $w \in S$, the graph $G(w)$ is *compatible* with $<_1$ and $<_2$ if for any $(a, b), (c, d) \in E(w)$, one has

$$a <_1 c \implies b \leq_2 d.$$

Example

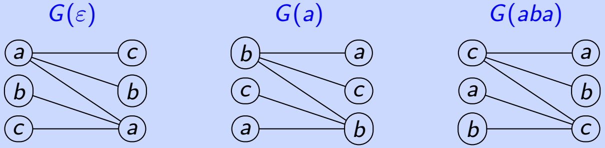
Let S be the Fibonacci set. Set $a <_1 b$ and $b <_2 a$.



We say that a biextendable set S is a *planar tree set* w.r.t. $<_1$ and $<_2$ on A if for any $w \in S$, the graph $G(w)$ is a tree compatible with $<_1$ and $<_2$.

Example

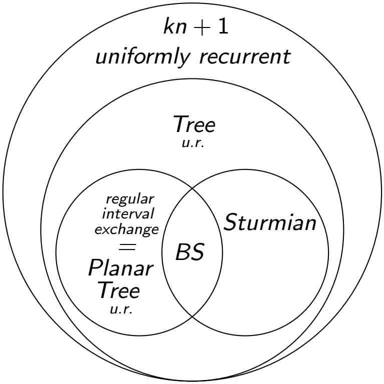
The *Tribonacci set* is not a planar tree set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



It is not possible to find two orders on A making the three graphs planar.

Theorem [Ferenczi, Zamboni (2008)]

A set S is a regular interval exchange set on A if and only if it is a uniformly recurrent planar tree set containing A.



Outline

Motivation

1. Two important classes
2. Acyclic, connected and tree sets
3. Bifix decoding
   - Bifix codes
   - Maximal bifix decoding

A set $X \subset A^+$ of nonempty words over an alphabet A is a *bifix code* if it does not contain any proper prefix or suffix of its elements.

Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
- $\{ac, bcc, bcbca\}$

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Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
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A bifix code $X \subset S$ is S-*maximal* if it is not properly contained in a bifix code $Y \subset S$.

Example

Let S be the Fibonacci set. The set $X = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an A^*-maximal bifix code, indeed $X \subset Y = X \cup \{bb\}$.

A *coding morphism* for a bifix code $X \subset A^+$ is a morphism $f : B^* \to A^*$ which maps bijectively B onto X.

Example

Let's consider the bifix code $X = \{aa, ab, ba\}$ on $A = \{a, b\}$ and let $B = \{u, v, w\}$. The map

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

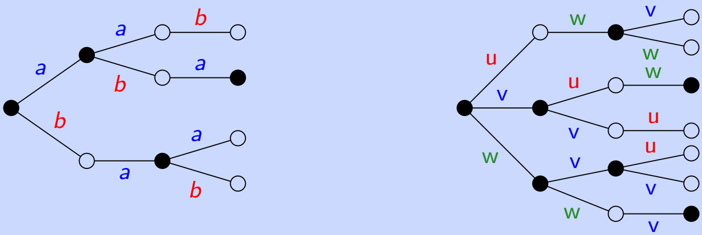
is a coding morphism for X.

If S is factorial and X is an S-maximal bifix code, we call the set $f^{-1}(S)$ a *maximal bifix decoding* of S.

Example

Let S be the Fibonacci set.
Let us consider the S-maximal bifix code $X = \{aa, ab, ba\}$ and the coding morphism

$$f : u \mapsto aa, \quad v \mapsto ab, \quad w \mapsto ba.$$

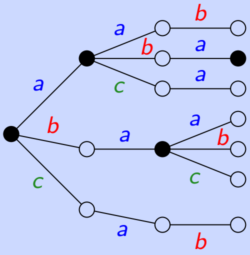


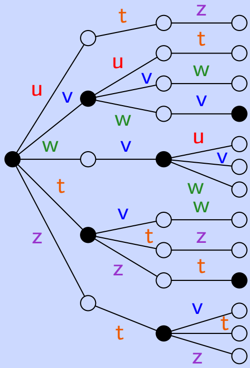
$f^{-1}(S)$ is not a Sturmian set. But it is a regular interval exchange sets (as S).

Example

Let T be the Tribonacci set.
Let us consider the T-maximal bifix code $X = \{aa, ab, ac, ba, ca\}$ and the coding morphism

$$g : u \mapsto aa, \quad v \mapsto ab, \quad w \mapsto ac, \quad t \mapsto ba, \quad z \mapsto ca.$$

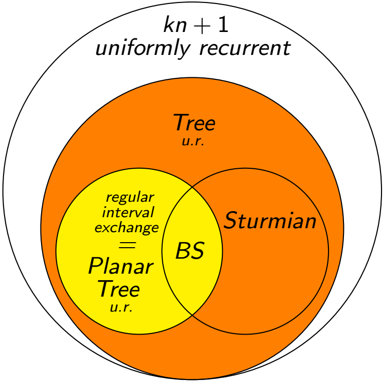




$g^{-1}(T)$ is not a Sturmian set. But it is a tree set (as T).

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

The family of uniformly recurrent tree sets is closed under maximal bifix decoding (and so is the family of u.r. planar tree sets).



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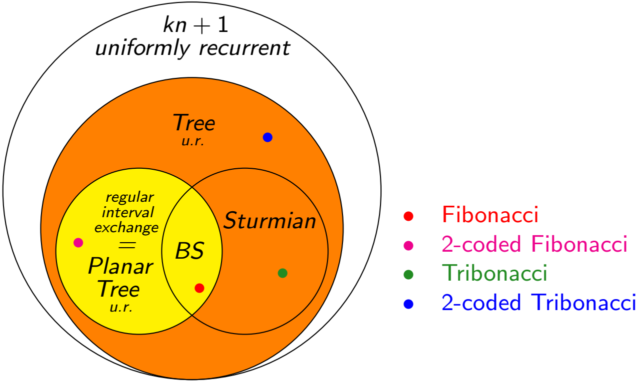

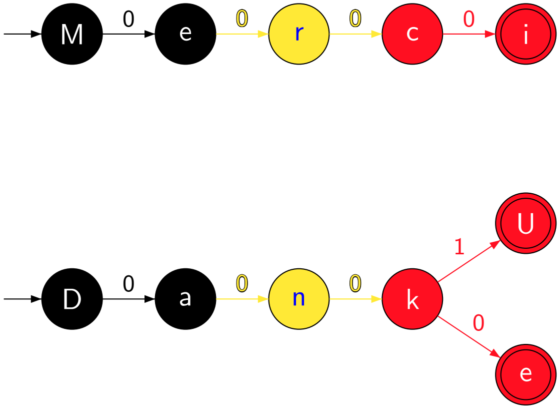
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